Suppose that $X$ is a nonempty set. Let $A$ denote the set of functions from $X$ to $X$. Let $B$ denote the set of $h \in A$ such that $h$ is bijective. Define a relation $R$ on $A$ by

$$R = \{(f, g) \in A \times A : \exists h \in B \text{ such that } h \circ f = g \circ h\}.$$ 

Observe that if $(f, g) \in A \times A$, $h \in B$ and $h \circ f = g \circ h$, then

$$h^{-1} \circ h \circ f = h^{-1} \circ g \circ h.$$

1. Prove that $R$ is an equivalence relation on $A$.

   **proof:**
   First, we prove that $R$ is reflexive. Suppose that $f \in A$. The identity function $i_X : X \to X$ is bijective, so $i_X \in B$. Moreover, $i_X \circ f = f \circ i_X$. It follows that $(f, f) \in R$. Since $f$ was arbitrary, $R$ is reflexive.

   Second, we prove that $R$ is symmetric. Suppose that $f, g \in A$, and $(f, g) \in R$. There is some $h \in B$ with $h \circ f = g \circ h$. Since $h$ is bijective, $h^{-1}$ exists and $h^{-1} \in B$. We have

   $$f = h^{-1} \circ h \circ f = h^{-1} \circ g \circ h.$$

   It follows that

   $$f \circ h^{-1} = h^{-1} \circ g \circ h \circ h^{-1} = h^{-1} \circ g.$$

   Thus, $(g, f) \in R$. Since $f$ and $g$ were arbitrary, $R$ is symmetric.

   Third, we prove that $R$ is transitive. Suppose that $f, g, k \in A$, $(f, g) \in R$, and $(g, k) \in R$. For some $h_1, h_2 \in B$ we have

   $$h_1 \circ f = g \circ h_1, \text{ and } h_2 \circ g = k \circ h_2.$$

   Set $h_3 = h_2 \circ h_1$. Then $h_3 \in B$, and

   $$h_3 \circ f = h_2 \circ h_1 \circ f = h_2 \circ g \circ h_1 = k \circ h_2 \circ h_1 = k \circ h_3.$$

   It follows that $(f, k) \in R$. Since $f, g$, and $k$ were arbitrary, $R$ is transitive.

   Finally, since $R$ is reflexive, transitive, and symmetric, $R$ is an equivalence relation.
2. Prove that if \((f, g) \in R\), then \((f \circ f, g \circ g) \in R\).

**proof:**
Suppose that \((f, g) \in R\). Then there is some \(h \in B\) with \(h \circ f = g \circ h\). We have

\[
h \circ f \circ f = g \circ h \circ f = g \circ g \circ h.
\]

Thus, \((f \circ f, g \circ g) \in R\).

3. Suppose that \((f, g) \in R\). Suppose that there is some \(x \in X\) with \(f(x) = x\). Prove that there is some \(y \in X\) with \(g(y) = y\).

**proof:**
Suppose that \((f, g) \in R\). Suppose that there is some \(x \in X\) with \(f(x) = x\). Since \((f, g) \in R\), there is some \(h \in B\) with \(h \circ f = g \circ h\).

Set \(y = h(x)\). Then \(y \in X\), and

\[
g(y) = g(h(x)) = h(f(x)) = h(x) = y.
\]