1. Suppose that \( \mathcal{F} \) and \( \mathcal{G} \) are families of sets. Prove that if \( \mathcal{F} \subseteq \mathcal{G} \) then \( \bigcup \mathcal{F} \subseteq \bigcup \mathcal{G} \).

**Proof:** Suppose that \( \mathcal{F} \subseteq \mathcal{G} \). Suppose that \( x \in \bigcup \mathcal{F} \). We can choose \( D \in \mathcal{F} \) such that \( x \in D \). Since \( \mathcal{F} \subseteq \mathcal{G} \) we have \( D \in \mathcal{G} \). Since \( x \in D \) and \( D \in \mathcal{G} \) it follows that \( x \in \bigcup \mathcal{G} \). Since \( x \) was arbitrary, we conclude that \( \bigcup \mathcal{F} \subseteq \bigcup \mathcal{G} \).

2. Prove that for any sets \( A, B, C \) and \( D \), if \( A \times B \) and \( C \times D \) are disjoint, then either \( A \) and \( C \) are disjoint or \( B \) and \( D \) are disjoint.

**Proof:** Suppose that \( A, B, C \) and \( D \) are sets. We prove the contrapositive. Suppose that it is not true that either \( A \) and \( C \) are disjoint or \( B \) and \( D \) are disjoint. Then we can choose \( x \in A \cap C \) and \( y \in B \cap D \). Then \( x \in A \), \( x \in C \), \( y \in B \), and \( y \in D \). Therefore \( (x, y) \in A \times B \) and \( (x, y) \in C \times D \), so \( A \times B \) and \( C \times D \) are not disjoint.

3. Suppose that \( R \) and \( S \) are transitive relations on \( A \). Prove that if \( S \circ R \subseteq R \circ S \) then \( R \circ S \) is transitive.

**Proof:** Suppose that \( R \) and \( S \) are transitive relations on \( A \). Suppose that \( S \circ R \subseteq R \circ S \). We have

\[
(R \circ S) \circ (R \circ S) = R \circ (S \circ R) \circ S \subseteq R \circ (R \circ S) \circ S = (R \circ R) \circ (S \circ S).
\]

Since \( R \) and \( S \) are transitive it follows from a previous theorem that \( (R \circ R) \subseteq R \) and \( (S \circ S) \subseteq S \). Hence \( (R \circ R) \circ (S \circ S) \subseteq R \circ S \). It follows that

\[
(R \circ S) \circ (R \circ S) \subseteq R \circ S.
\]

By a previous theorem we conclude that \( R \circ S \) is transitive.

4. Suppose \( R \) is a relation from \( A \) to \( B \) and \( S \) and \( T \) are relations from \( B \) to \( C \). Prove that

\[
(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R.
\]

**Proof:** Suppose that \( (x, z) \in (S \circ R) \setminus (T \circ R) \). Then \( (x, z) \in (S \circ R) \), and \( (x, z) \notin (T \circ R) \). Since \( (x, z) \in (S \circ R) \), we can choose \( y \in B \) such that \( (x, y) \in R \) and \( (y, z) \in S \).

We claim that \( (y, z) \notin T \). We prove this claim by contradiction. Suppose that \( (y, z) \in T \). Since \( (x, y) \in R \) we have \( (x, z) \in (T \circ R) \). This is a contradiction. This proves the claim.

We now have \( (x, y) \in R \) and \( (y, z) \in S \setminus T \). It follows that \( (x, z) \in (S \setminus T) \circ R \). Since \( (x, z) \) was arbitrary, we have

\[
(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R.
\]
5. Prove that $\forall x \in \mathbb{R}[(\exists y \in \mathbb{R} \ (x + y = xy)) \iff x \neq 1]$.

**Proof:** Suppose that $x \in \mathbb{R}$.

First, suppose that $x \neq 1$. Set $y_0 = x/(x - 1)$ and observe that

$$xy_0 = \frac{x^2}{x - 1} = x + \frac{x}{x - 1} = x + y_0.$$ 

Therefore if $x \neq 1$ then there does exist $y \in \mathbb{R}$ such that $x + y = xy$. On the other hand, if $x = 1$ then the equation $x + y = xy$ reduces to $1 + y = y$, which has no solution in $y$. Therefore there exists $y \in \mathbb{R}$ such that $x + y = xy$ if and only if $x \neq 1$.

6. Prove that for any sets $A$ and $B$, if $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$.

**Proof:** Suppose that $A$ and $B$ are sets and $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$. Since $(A \cup B) \in \mathcal{P}(A \cup B)$, it follows that $(A \cup B) \in \mathcal{P}(A) \cup \mathcal{P}(B)$. So either $(A \cup B) \in \mathcal{P}(A)$ or $(A \cup B) \in \mathcal{P}(B)$. In the first case, $B \subseteq A$ while in the second case $A \subseteq B$. So in each possible case either $A \subseteq B$ or $B \subseteq A$.

7. Suppose that $f : A \to B$ and $g : B \to C$. Prove that if $f$ is onto and $g$ is not one-to-one, then $g \circ f$ is not one-to-one.

**Proof:** Suppose that $f$ is onto and $g$ is not one-to-one. Since $g$ is not one-to-one, we can choose $b_1, b_2 \in B$ with $b_1 \neq b_2$ and $g(b_1) = g(b_2)$. Since $f$ is onto, we can choose $a_1, a_2 \in A$ with $f(a_1) = b_1$ and $f(a_2) = b_2$. It follows, since $f$ is a function, that $a_1 \neq a_2$. We have

$$(g \circ f)(a_1) = g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2)) = (g \circ f)(a_2).$$

Hence, $g \circ f$ is not one-to-one.

8. Let $U$ be any set. Prove that there is a unique $A \in \mathcal{P}(U)$ such that for every $B \in \mathcal{P}(U)$, $A \cap B = B$.

**Answer:**

First, we prove existence. Set $A = U$. Suppose that $B \in \mathcal{P}(U)$. Then $B \subseteq U$. It follows that

$$A \cap B = U \cap B = B.$$ 

Next, we prove uniqueness. Suppose that $A \in \mathcal{P}(U)$ and for every $B \in \mathcal{P}(U)$ we have $A \cap B = B$. Since $U \in \mathcal{P}(U)$, we have $A \cap U = U$. On the other hand, as $A \subseteq U$, we have that $A \cap U = A$. It follows that $A = U$.

9. Let $A = \mathbb{R} \setminus \{2\}$, and let $f$ be the function with domain $A$ defined by the formula $f(x) = \frac{3x}{x - 2}$. Prove that $f$ is a one-to-one, onto function from $A$ to some set $B \subseteq \mathbb{R}$. Find $B$. Also, find a formula for $f^{-1}$.
10. Prove the following by mathematical induction: For all $n \in \mathbb{N}$, \( 9 \mid (4^n + 6n - 1) \).

**Proof:** For $n \in \mathbb{N}$ we let $P(n)$ be the statement $9 \mid (4^n + 6n - 1)$. We use induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$. First, $P(0)$ says that 9 divides $4^0 + 6 \cdot 0 - 1 = 0$, which is true. Now let $n \in \mathbb{N}$ and assume that $P(n)$ is true. Then for some $q \in \mathbb{Z}$ we have $4^n + 6n - 1 = 9q$. Hence $4^n = 9q - 6n + 1$, so $4^{n+1} = 4 \cdot 4^n = 36q - 24n + 4$. It follows that $4^{n+1} + 6(n+1) - 1 = 36q - 24n + 4 + 6n + 6 - 1 = 36q - 18n + 9 = 9(4q - 2n + 1)$.

Since $4q - 2n + 1 \in \mathbb{Z}$ we see that $9 \mid (4^{n+1} + 6(n+1) - 1)$. Therefore $P(n+1)$ is true. It follows by induction that $P(n)$ is true for all $n \in \mathbb{N}$. 

Proof: First, we claim that $3 \notin \text{Ran}(f)$. We prove this by contradiction. Suppose that $3 \in \text{Ran}(f)$. Then for some real number $x \in A$, we have $3 = \frac{3x}{x-2}$. It follows that $3x - 6 = 3x$, a contradiction. This proves the claim.

Set $B = \mathbb{R} \setminus \{3\}$. It follows from the claim that $f : A \rightarrow B$.

We define a function $g$ on $B$ by the formula $g(y) = \frac{2y}{y-3}$. We claim that $2 \notin \text{Ran}(g)$.

We prove this by contradiction. Suppose that $2 \in \text{Ran}(g)$. Then for some real number $y \in B$, we have $2 = \frac{2y}{y-3}$. It follows that $2y - 6 = 2y$, a contradiction. This proves the claim. It follows from the claim that $g : B \rightarrow A$.

Next, we prove that $g \circ f = i_A$. We have that both $g \circ f$ and $i_A$ are functions from $A$ to $A$. Suppose that $x \in A$. Then

$$ (g \circ f)(x) = g(f(x)) = g\left(\frac{3x}{x-2}\right) = \frac{2 \cdot \frac{3x}{x-2}}{x-2} = \frac{6x}{3x - 3x + 6} = x. $$

It follows that $g \circ f = i_A$.

Finally, we prove that $f \circ g = i_B$. We have that both are functions from $B$ to $B$. Suppose that $y \in B$. Then

$$ (f \circ g)(y) = f(g(y)) = f\left(\frac{2y}{y-3}\right) = \frac{3 \cdot \frac{2y}{y-3}}{2 - \frac{2y}{y-3} - 2} = \frac{6y}{2y - 2y + 6} = y. $$

It follows that $f \circ g = i_B$.

Now, we have $f : A \rightarrow B$, $g : B \rightarrow A$, $g \circ f = i_A$, and $f \circ g = i_B$. It follows from a previous theorem that $g = f^{-1}$, and thus a formula for $f^{-1}$ is given by $f^{-1}(y) = \frac{2y}{y-3}$. Since $f^{-1}$ is a function, it follows from a previous theorem that $f$ is a one-to one, onto function from $A$ to $B$. 

10. Prove the following by mathematical induction: For all $n \in \mathbb{N}$,

$$ 9 \mid (4^n + 6n - 1). $$

Recall that this notation means that $(4^n + 6n - 1)$ is an integer multiple of 9.

**Proof:** For $n \in \mathbb{N}$ we let $P(n)$ be the statement $9 \mid (4^n + 6n - 1)$. We use induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$. First, $P(0)$ says that 9 divides $4^0 + 6 \cdot 0 - 1 = 0$, which is true. Now let $n \in \mathbb{N}$ and assume that $P(n)$ is true. Then for some $q \in \mathbb{Z}$ we have $4^n + 6n - 1 = 9q$. Hence $4^n = 9q - 6n + 1$, so $4^{n+1} = 4 \cdot 4^n = 36q - 24n + 4$. It follows that $4^{n+1} + 6(n+1) - 1 = 36q - 24n + 4 + 6n + 6 - 1 = 36q - 18n + 9 = 9(4q - 2n + 1)$.

Since $4q - 2n + 1 \in \mathbb{Z}$ we see that $9 \mid (4^{n+1} + 6(n+1) - 1)$. Therefore $P(n+1)$ is true. It follows by induction that $P(n)$ is true for all $n \in \mathbb{N}$. 
