MHF 3202, Dr. Block, Sample Exam 1 with answers, Spring 2020

There are 7 problems worth a total of 40 points.

1. (5 points) Write the following set by listing the elements of the set between braces.

   $\mathscr{P}(\{1,2\}) \times \mathscr{P}(\{3\})$

   Answer:

   $$\{(\phi,\phi),(\phi,\{3\}),(\{1\},\phi),(\{1\},\{3\}),(\{2\},\phi),(\{2\},\{3\}),(\{1,2\},\phi),(\{1,2\},\{3\})\}$$

2. (6 points) Construct a truth table for the formula $(P \wedge \sim Q) \Rightarrow R$.

   Answer:

P	Q	R	$\sim Q$	$P\wedge \sim Q$	$(P\wedge \sim Q) \Rightarrow R$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

3. (6 points) Negate the following sentence.

   For every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x-5| < \delta$, then $|x^2 - 25| < \epsilon$.

   Answer: There exists $\epsilon > 0$ such that for all $\delta > 0$ there exists x with $|x-5| < \delta$ and $|x^2-25| \geq \epsilon$.

4. (2 points) Determine whether the following statement is true or false.

   $$\exists a \in \mathbb{Z},\ \forall b \in \mathbb{Z},\ a+b=0$$

   Answer: False.

5. (7 points) Prove the following using a direct proof.

   If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$.

   Proof: Suppose that $x \in \mathbb{R}$ and $0 < x < 4$. Since the square of any real number is greater than or equal to zero, we have $(x-2)^2 \geq 0$. By algebra, we obtain $x^2 - 4x + 4 \geq 0$. Adding -4 to each side of the inequality, we have $x^2 - 4x \geq -4$. Now, multiplying each side by -1 and factoring, we obtain

$$x(4-x) \leq 4.$$

   Since $0 < x < 4$, using order properties of $\mathbb{R}$ we have $\frac{1}{x(4-x)} > 0$. So multipying each side of the displayed inequality by $\frac{1}{x(4-x)}$, we obtain $1 \leq \frac{4}{x(4-x)}$. Therefore, $\frac{4}{x(4-x)} \geq 1$.

6. (7 points) Prove the following with contrapositive proof.

   Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.

   Proof: Suppose that a is not odd. Then a is even. So for some integer k we have $a = 2k$. It follows that

$$a^2 = 4k^2 = 4j,$$

   where $j = k^2 \in \mathbb{Z}$. Therefore, a^2 is divisible by 4.

7. (7 points) Prove the following using either direct proof or contrapositive proof.

   If $a \in \mathbb{Z}$ and $a \equiv 1 \pmod{5}$, then $a^2 \equiv 1 \pmod{5}$.

   Proof: (direct proof) Suppose that $a \in \mathbb{Z}$ and $a \equiv 1 \pmod{5}$. By definition, $5|(a-1)$. So for some integer k we have $a - 1 = 5k$. It follows that

$$a^2 - 1 = (a+1)(a-1) = (a+1)(5k) = 5j,$$

   where $j = ak + k \in \mathbb{Z}$. Thus, $5|(a^2-1)$. Therefore, $a^2 \equiv 1 \pmod{5}$.