MHF 3202: Sample Proof

Problem 18 page 134 Suppose that \( F \) and \( G \) are families of sets. Prove that 
\[
(\bigcup F) \cap (\bigcup G) \subseteq \bigcup (F \cap G) \iff \forall A \in F \forall B \in G ((A \cap B) \subseteq \bigcup (F \cap G))
\]

Proof. 
First, suppose that 
\[
(\bigcup F) \cap (\bigcup G) \subseteq \bigcup (F \cap G)
\]

Suppose that \( A \in F \) and \( B \in G \).
Suppose that \( x \in A \cap B \).
Then \( x \in (\bigcup F) \cap (\bigcup G) \). It follows that \( x \in \bigcup (F \cap G) \). Since \( x \) was arbitrary, we conclude that \( (A \cap B) \subseteq \bigcup (F \cap G) \). Since \( A \) and \( B \) were arbitrary, we conclude that 
\[
\forall A \in F \forall B \in G ((A \cap B) \subseteq \bigcup (F \cap G))
\]

Second, suppose that 
\[
\forall A \in F \forall B \in G ((A \cap B) \subseteq \bigcup (F \cap G))
\]

Suppose that \( x \in (\bigcup F) \cap (\bigcup G) \). Then for some \( A \in F \) we have \( x \in A \), and for some \( B \in G \) we have \( x \in B \). Hence, \( x \in A \cap B \). It follows that \( x \in \bigcup (F \cap G) \). Since \( x \) was arbitrary, we conclude that \( (\bigcup F) \cap (\bigcup G) \subseteq \bigcup (F \cap G) \).

Finally, (from the two implications proved above) we conclude that 
\[
(\bigcup F) \cap (\bigcup G) \subseteq \bigcup (F \cap G) \iff \forall A \in F \forall B \in G ((A \cap B) \subseteq \bigcup (F \cap G))
\]