MHF 3202: Solutions to Problem Set 7

(a) First, we prove that $S$ is reflexive. Suppose that $x \in B$. Since $B \subseteq A$, we have $x \in A$. As $R$ is reflexive, it follows that $(x,x) \in R$. Thus, $(x,x) \in R \cap (B \times B) = S$.

Next, we prove that $S$ is symmetric. Suppose that $(x,y) \in S$. Then $(x,y) \in R$, $x \in B$, and $y \in B$. Since $R$ is symmetric, $(y,x) \in R$. It follows that $(y,x) \in R \cap (B \times B) = S$.

Finally, we prove that $S$ is transitive. Suppose that $(x,y) \in S$ and $(y,z) \in S$. Then $x,y,z \in B$, $(x,y) \in R$ and $(y,z) \in R$. Since $R$ is transitive, $(x,z) \in R$. Thus, $(x,z) \in R \cap (B \times B) = S$.

Since $S$ is reflexive, symmetric, and transitive, it follows that $S$ is an equivalence relation on $B$.

(b) Suppose that $x \in B$.

First, we prove that $[x]_S \subseteq [x]_R \cap B$. Suppose that $y \in [x]_S$. Then $(x,y) \in S$. It follows that $y \in B$, and $(x,y) \in R$. Hence, $y \in [x]_R \cap B$.

Second, we prove that $[x]_R \cap B \subseteq [x]_S$. Suppose that $y \in [x]_R \cap B$. Then $y \in B$, and $(x,y) \in R$. Since also $x \in B$, it follows that $(x,y) \in R \cap (B \times B) = S$. Hence $y \in [x]_S$. 