MHF 3202, Dr. Block, Chapter 4 and 5 Notes

1. In these notes we suppose that \( R \) is a relation on a set \( A \). Also, recall that \( xRy \) means \((x, y) \in R\). We summarize some basic definitions.

2. \( R \) is reflexive iff \( \forall x \in A (xRx) \).

3. \( R \) is symmetric iff \( \forall x \in A, \forall y \in A, (xRy \rightarrow yRx) \).

4. \( R \) is transitive iff \( \forall x \in A \forall y \in A, \forall z \in A, ((xRy \land yRz) \rightarrow xRz) \).

5. \( R \) is antisymmetric iff \( \forall x \in A \forall y \in A, ((xRy \land yRx) \rightarrow x = y) \).

6. \( R \) is a partial order iff \( R \) is reflexive, transitive, and antisymmetric.

7. \( R \) is a total order iff \( R \) is a partial order and also the following holds:
   \[ \forall x \in A \forall y \in A (xRy \lor yRx) \).

8. Suppose that \( R \) is a partial order on \( A \). Suppose that \( B \subseteq A \) and \( b \in B \). We say that \( b \) is a smallest (or \( R \)-smallest) element of \( B \) iff \( \forall x \in B (bRx) \).

9. Suppose that \( R \) is a partial order on \( A \). Suppose that \( B \subseteq A \) and \( b \in B \). We say that \( b \) is a minimal (or \( R \)-minimal) element of \( B \) iff
   \[ \neg \exists x \in B (xRb \land x \neq b) \).

10. Suppose that \( R \) is a partial order on \( A \). Suppose that \( B \subseteq A \) and \( a \in A \). We say that \( a \) is a lower bound for \( B \) iff
    \[ \forall x \in B (aRx) \).

11. Suppose that \( R \) is a partial order on \( A \). Suppose that \( B \subseteq A \) and \( a \in A \). We say that \( a \) is an upper bound for \( B \) iff
    \[ \forall x \in B (xRa) \).

12. Suppose that \( R \) is a partial order on \( A \), and \( B \subseteq A \). Let \( L \) be the set of all lower bounds for \( B \). If \( L \) has a largest element, then this largest element is called the greatest lower bound of \( B \).
13. Suppose that $R$ is a partial order on $A$, and $B \subseteq A$. Let $U$ be the set of all upper bounds for $B$. If $U$ has a smallest element, then this smallest element is called the least upper bound of $B$.

14. Suppose that $A$ is a set and $\mathcal{F} \subseteq \mathcal{P}(A)$. We say that $\mathcal{F}$ is pairwise disjoint iff every pair of distinct elements of $\mathcal{F}$ are disjoint. We say that $\mathcal{F}$ is a partition of $A$ iff $\mathcal{F}$ is pairwise disjoint, $\bigcup \mathcal{F} = A$, and $\emptyset \notin \mathcal{F}$.

15. Suppose that $R$ is a relation on a set $A$. We say that $R$ is an equivalence relation iff $R$ is reflexive, transitive, and symmetric.

16. Suppose that $R$ is an equivalence relation on $A$. Suppose that $x \in A$. The equivalence class of $x$ denoted $[x]$ is given by

$$[x] = \{y \in A | xRy\}.$$ We let $A/R$ (in words, $A$ modulo $R$) denote the set of equivalence classes.

17. We have the following theorem: Suppose that $R$ is an equivalence relation on $A$. Then $A/R$ is a partition of $A$.

18. We also have the following theorem: Suppose that $A$ is a set, and $\mathcal{F}$ is a partition of $A$. Then there is an equivalence relation $R$ on $A$ such that $A/R = \mathcal{F}$.

19. Suppose that $F$ is a relation from $A$ to $B$. We say that $F$ is a function from $A$ to $B$ iff for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in F$. We use the notation $F : A \to B$ to indicate that $F$ is a function from $A$ to $B$. Also, if $a \in A$, we let $f(a)$ denote the unique $b \in B$ such that $(a, b) \in F$.

20. We have the following theorem: Suppose that $f$ and $g$ are functions from $A$ to $B$. Then $f = g$ if and only if $\forall a \in A (f(a) = g(a))$.

21. We have the following theorem: Suppose that $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ and for every $a \in A$ we have $(g \circ f)(a) = g(f(a))$. 
22. Suppose that \( f : A \to B \). We say that \( f \) is one-to-one iff for all \( a_1 \in A \) and \( a_2 \in A \) if \( f(a_1) = f(a_2) \) then \( a_1 = a_2 \). We say that \( f \) is onto iff for every \( b \in B \) there exists \( a \in A \) with \( f(a) = b \). Note that \( f \) is onto if and only if \( B \) is the range of \( f \).

23. Suppose that \( f : A \to B \). Then \( f \) is also a relation from \( A \) to \( B \). So the inverse relation \( f^{-1} \) is defined and is a relation from \( B \) to \( A \). We have the following theorem: \( f^{-1} \) is a function from \( B \) to \( A \) if and only if \( f \) is one-to-one and onto.

24. We have the following theorem: Suppose that \( f : A \to B \) and \( g : B \to A \). Suppose also that \( g \circ f = i_A \) and \( f \circ g = i_B \). Then \( g = f^{-1} \).