1. Definition and Axiom. Suppose that $A$ and $B$ are sets. We let $A \times B$ denote the set of ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$. Two ordered pairs $(c, d)$ and $(v, w)$ are equal if and only if $c = v$ and $v = w$. Any subset $R$ of $A \times B$ is called a relation from $A$ to $B$.

2. Theorem. Suppose that $A, B, C, D$ are sets.
   1. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
   2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
   3. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
   4. $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.
   5. $A \times \emptyset = \emptyset \times A = \emptyset$.

3. Definition. Suppose that $R$ is a relation from $A$ to $B$. We define the domain and range of $R$ as follows:
   \[ \text{Dom}(R) = \{a \in A | \exists b \in B ((a, b) \in R)\}. \]
   \[ \text{Ran}(R) = \{b \in B | \exists a \in A ((a, b) \in R)\}. \]

   Also, we define the inverse relation $R^{-1}$ from $B$ to $A$ by
   \[ R^{-1} = \{(y, x) \in B \times A | (x, y) \in R\}. \]

4. Definition. Suppose that $R$ is a relation from $A$ to $B$, and $S$ is a relation from $B$ to $C$. The composition of $S$ and $R$ is the relation from $A$ to $C$ given by
   \[ S \circ R = \{(a, c) \in A \times C | \exists b \in B ((a, b) \in R \land (b, c) \in S)\}. \]

5. Theorem. Suppose that $R$ is a relation from $A$ to $B$, and $S$ is a relation from $B$ to $C$. Then
   \[ (S \circ R)^{-1} = R^{-1} \circ S^{-1}. \]

6. Theorem. Suppose that $R$ is a relation from $A$ to $B$, $S$ is a relation from $B$ to $C$, and $T$ is a relation from $C$ to $D$. Then
   \[ (T \circ S) \circ R = T \circ (S \circ R). \]
7. Definition and Notation. If $A$ is a set and $R$ is a relation from $A$ to $A$, we say that $R$ is a relation on the set $A$. From now in these notes, we suppose that $R$ is a relation on the set $A$. Also, we sometimes use the notation $xRy$ instead of $(x, y) \in R$.

8. Theorem. Suppose that $R, S, T$ are relations on $A$. Suppose that $S \subseteq R$. Then $S \circ T \subseteq R \circ T$ and $T \circ S \subseteq T \circ R$.

9. Definition. $R$ is reflexive iff for all $\forall x \in A (xRx)$.

$R$ is symmetric iff $\forall x \in A, \forall y \in A (xRy \rightarrow yRx)$.

$R$ is transitive iff $\forall x \in A \forall y \in A, \forall z \in A, ((xRy \land yRz) \rightarrow xRz)$.

$R$ is antisymmetric iff $\forall x \in A \forall y \in A, ((xRy \land yRx) \rightarrow x = y)$.

10. Definition. The identity relation on $A$ is given by

$$i_A = \{(x, y) \in A \times A | y = x\}.$$ 

11. Theorem.

$R$ is reflexive iff $i_A \subseteq R$.

$R$ is symmetric iff $R = R^{-1}$.

$R$ is transitive iff $R \circ R \subseteq R$.

12. Definition. $R$ is a partial order iff $R$ is reflexive, transitive, and antisymmetric.

13. Definition. $R$ is a total order iff $R$ is a partial order and also the following holds:

$$\forall x \in A \forall y \in A (xRy \lor yRx).$$

14. Definition. Suppose that $R$ is a partial order on $A$. Suppose that $B \subseteq A$ and $b \in B$. We say that $b$ is a smallest (or $R$-smallest) element of $B$ iff $\forall x \in B (bRx)$.

15. Theorem. Suppose that $R$ is a partial order on $A$ and $B \subseteq A$. If $B$ has a smallest element, then this smallest element is unique.
16. Definition. Suppose that $R$ is a partial order on $A$. Suppose that $B \subseteq A$ and $b \in B$. We say that $b$ is a minimal (or $R$-minimal) element of $B$ iff 

$$\neg \exists x \in B(xRb \land x \neq b).$$

17. Theorem. Suppose that $R$ is a partial order on $A$ and $B \subseteq A$. Suppose that $b$ is the smallest element of $B$. Then $b$ is also the unique minimal element of $B$.

18. Theorem. Suppose that $R$ is a total order on $A$ and $B \subseteq A$. Suppose that $b$ is a minimal element of $B$. Then $b$ is also the smallest element of $B$.

19. Definition. Suppose that $R$ is a partial order on $A$. Suppose that $B \subseteq A$ and $a \in A$. We say that $a$ is a lower bound for $B$ iff 

$$\forall x \in B(aRx).$$

20. Definition. Suppose that $R$ is a partial order on $A$. Suppose that $B \subseteq A$ and $a \in A$. We say that $a$ is an upper bound for $B$ iff 

$$\forall x \in B(xRa).$$

21. Definition. Suppose that $R$ is a partial order on $A$, and $B \subseteq A$. Let $L$ be the set of all lower bounds for $B$. If $L$ has a largest element, then this largest element is called the greatest lower bound of $B$.

22. Definition. Suppose that $R$ is a partial order on $A$, and $B \subseteq A$. Let $U$ be the set of all upper bounds for $B$. If $U$ has a smallest element, then this smallest element is called the least upper bound of $B$.

23. Definition. Suppose that $A$ is a set and $\mathcal{F} \subseteq \mathcal{P}(A)$. We say that $\mathcal{F}$ is pairwise disjoint iff every pair of distinct elements of $\mathcal{F}$ are disjoint. We say that $\mathcal{F}$ is a partition of $A$ iff $\mathcal{F}$ is pairwise disjoint, $\bigcup \mathcal{F} = A$, and $\emptyset \notin \mathcal{F}$.

24. Definition. $R$ is an equivalence relation iff $R$ is reflexive, transitive, and symmetric.
25. Definition. Suppose that $R$ is an equivalence relation on $A$. Suppose that $x \in A$. The equivalence class of $x$ denoted $[x]$ is given by

\[ [x] = \{ y \in A | xRy \}. \]

We let $A/R$ (in words, $A$ modulo $R$) denote the set of equivalence classes.

26. Theorem. Suppose that $R$ is an equivalence relation on $A$. Suppose that $x, y \in A$. Then $xRy$ iff $[x] = [y]$.

27. Theorem. Suppose that $R$ is an equivalence relation on $A$. Then $A/R$ is a partition of $A$.

28. Theorem. Suppose that $A$ is a set, and $\mathcal{F}$ is a partition of $A$. Then there is an equivalence relation $R$ on $A$ such that $A/R = \mathcal{F}$.

29. Definition. We let $\mathbb{Z}$ denote the set of integers,

\[ \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}. \]

30. Theorem. The sum of two integers is an integer. The product of two integers is an integer. The additive inverse of an integer is an integer.

31. Theorem. (Division Algorithm) If $a$ and $b$ are integers with $a > 0$, then there exist unique integers $q$ and $r$ such that $b = qa + r$ and $0 \leq r < a$.

32. Definition. Let $a$ and $b$ be integers. We say that $a$ divides $b$ iff there exists an integer $c$ such that $b = ac$.

33. Definition. Let $m \geq 2$ be an integer, and let $x$ and $y$ be integers. We say that $x$ is congruent to $y$ modulo $m$ iff $m$ divides $x - y$. We use the notation,

\[ x \equiv y \pmod{m}. \]

34. Theorem. Let $m \geq 2$ be an integer, and let $C_m$ denote the set of ordered pairs $(x, y) \in (\mathbb{Z} \times \mathbb{Z})$ such that

\[ x \equiv y \pmod{m}. \]
Then $C_m$ is an equivalence relation. Moreover, there are exactly $m$ distinct equivalence classes given by $[0], [1], \ldots, [m-1]$.

35. Remark and Definition. Consider the case $m = 2$ in the previous theorem. There are two distinct equivalence classes, $[0], [1]$. Integers in $[0]$ are called even. Integers in $[1]$ are called odd.

36. Definition: Suppose that $F$ is a relation from $A$ to $B$. We say that $F$ is a function from $A$ to $B$ iff for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in F$. We use the notation $F : A \rightarrow B$ to indicate that $F$ is a function from $A$ to $B$. Also, if $a \in A$, we let $F(a)$ denote the unique $b \in B$ such that $(a, b) \in F$.

37. Theorem: Suppose that $f$ and $g$ are functions from $A$ to $B$. Then $f = g$ if and only if $\forall a \in A (f(a) = g(a))$.

38. Theorem: Suppose that $f : A \rightarrow B$ and $g : B \rightarrow C$. Then $g \circ f : A \rightarrow C$ and for every $a \in A$ we have $(g \circ f)(a) = g(f(a))$.

39. Definition and Remark. Suppose that $f : A \rightarrow B$. We say that $f$ is one-to-one iff for all $a_1 \in A$ and $a_2 \in A$ if $f(a_1) = f(a_2)$ then $a_1 = a_2$. We say that $f$ is onto iff for every $b \in B$ there exists $a \in A$ with $f(a) = b$. Note that $f$ is onto if and only if $B$ is the range of $f$.

40. Remark. Suppose that $f : A \rightarrow B$. Then $f$ is one-to-one and onto iff for all $b \in B$ there is a unique $a \in A$ with $f(a) = b$.

41. Theorem: Suppose that $f : A \rightarrow B$ and $g : B \rightarrow C$. If $f$ and $g$ are one-to-one, then $g \circ f$ is one-to-one. If $f$ and $g$ are onto, then $g \circ f$ is onto.

42. Remark and Theorem. Suppose that $f : A \rightarrow B$. Then $f$ is also a relation from $A$ to $B$. So the inverse relation $f^{-1}$ is defined and is a relation from $B$ to $A$. We have the following theorem: $f^{-1}$ is a function from $B$ to $A$ if and only if $f$ is one-to-one and onto.

43. Theorem: Suppose that $f : A \rightarrow B$ and $g : B \rightarrow A$. Suppose also that $g \circ f = i_A$ and $f \circ g = i_B$. Then $g = f^{-1}$. 
44. Axiom. (Well-ordering principle). Every nonempty subset of \( \mathbb{N} \) has a smallest element.

45. Theorem. (Mathematical Induction). Suppose \( j \in \mathbb{N} \). Suppose that 
\( P(x) \) is a statement with a free variable. Suppose that
1. \( P(j) \) and 
2. For all \( k \in \mathbb{N} \) with \( k \geq j \) if \( P(k) \) holds then \( P(k + 1) \) also holds.
Then for all \( n \in \mathbb{N} \) with \( n \geq j \) we have \( P(n) \).

46. Remark. Similar to Mathematical Induction, we sometimes use recursive definitions. We may define a function \( f \) with domain \( \mathbb{N} \) by defining \( f(0) \) and defining \( f(k + 1) \) in terms of \( f(k) \). For example, if \( x \) is a real number we may define \( x^n \) by 
\( x^0 = 1 \) and \( x^{(k+1)} = xx^k \).

47. Theorem. (Mathematical Induction, Strong Form). Suppose \( j \in \mathbb{N} \). Suppose that \( P(x) \) is a statement with a free variable. Suppose that
1. \( P(j) \) and 
2. For all \( k \in \mathbb{N} \) with \( k \geq j \) if \( P(s) \) holds for all \( s \in \mathbb{N} \) with \( j \leq s \leq k \) then \( P(k + 1) \) also holds.
Then for all \( n \in \mathbb{N} \) with \( n \geq j \) we have \( P(n) \).