1. Definition and Axiom. Suppose that $A$ and $B$ are sets. We let $A \times B$ denote the set of ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$. Two ordered pairs $(c, d)$ and $(v, w)$ are equal if and only if $c = v$ and $d = w$. Any subset $R$ of $A \times B$ is called a relation from $A$ to $B$.

2. Theorem. Suppose that $A, B, C, D$ are sets.
   1. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
   2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
   3. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
   4. $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.
   5. $A \times \emptyset = \emptyset \times A = \emptyset$.

3. Definition. Suppose that $R$ is a relation from $A$ to $B$. We define the domain and range of $R$ as follows:
   $$\text{Dom}(R) = \{a \in A | \exists b \in B ((a, b) \in R)\}.$$
   $$\text{Ran}(R) = \{b \in B | \exists a \in A ((a, b) \in R)\}.$$
   Also, we define the inverse relation $R^{-1}$ from $B$ to $A$ by
   $$R^{-1} = \{(y, x) \in B \times A | (x, y) \in R\}.$$

4. Definition. Suppose that $R$ is a relation from $A$ to $B$, and $S$ is a relation from $B$ to $C$. The composition of $S$ and $R$ is the relation from $A$ to $C$ given by
   $$S \circ R = \{(a, c) \in A \times C | \exists b \in B ((a, b) \in R \land (b, c) \in S)\}.$$

5. Theorem. Suppose that $R$ is a relation from $A$ to $B$, and $S$ is a relation from $B$ to $C$. Then
   $$(S \circ R)^{-1} = R^{-1} \circ S^{-1}.$$

6. Theorem. Suppose that $R$ is a relation from $A$ to $B$, $S$ is a relation from $B$ to $C$, and $T$ is a relation from $C$ to $D$. Then
   $$(T \circ S) \circ R = T \circ (S \circ R).$$
7. Definition and Notation. If $A$ is a set and $R$ is a relation from $A$ to $A$, we say that $R$ is a relation on the set $A$. From now in these notes, we suppose that $R$ is a relation on the set $A$. Also, we sometimes use the notation $xRy$ instead of $(x, y) \in R$.

8. Theorem. Suppose that $R, S, T$ are relations on $A$. Suppose that $S \subseteq R$. Then $S \circ T \subseteq R \circ T$ and $T \circ S \subseteq T \circ R$.

9. Definition. $R$ is reflexive iff for all $\forall x \in A(xRx)$.

$R$ is symmetric iff $\forall x, y \in A, (xRy \rightarrow yRx)$.

$R$ is transitive iff $\forall x, y, z \in A, ((xRy \land yRz) \rightarrow xRz)$.

$R$ is antisymmetric iff $\forall x, y \in A, ((xRy \land yRx) \rightarrow x = y)$.

10. Definition. The identity relation on $A$ is given by

$$i_A = \{(x, y) \in A \times A | y = x\}.$$ 

11. Theorem.

$R$ is reflexive iff $i_A \subseteq R$.

$R$ is symmetric iff $R = R^{-1}$.

$R$ is transitive iff $R \circ R \subseteq R$.

12. Definition. $R$ is a partial order iff $R$ is reflexive, transitive, and antisymmetric.

13. Definition. $R$ is a total order iff $R$ is a partial order and also the following holds:

$$\forall x \in A \forall y \in A (xRy \lor yRx).$$

14. Definition. Suppose that $R$ is a partial order on $A$. Suppose that $B \subseteq A$ and $b \in B$. We say that $b$ is a smallest (or $R$-smallest) element of $B$ iff $\forall x \in B (bRx)$.

15. Theorem. Suppose that $R$ is a partial order on $A$ and $B \subseteq A$. If $B$ has a smallest element, then this smallest element is unique.
16. Definition. Suppose that $R$ is a partial order on $A$. Suppose that $B \subseteq A$ and $b \in B$. We say that $b$ is a minimal (or $R$-minimal) element of $B$ iff
\[-\exists x \in B(xRb \land x \neq b)\].

17. Theorem. Suppose that $R$ is a partial order on $A$ and $B \subseteq A$. Suppose that $b$ is the smallest element of $B$. Then $b$ is also the unique minimal element of $B$.

18. Theorem. Suppose that $R$ is a total order on $A$ and $B \subseteq A$. Suppose that $b$ is a minimal element of $B$. Then $b$ is also the smallest element of $B$.

19. Definition. Suppose that $R$ is a partial order on $A$. Suppose that $B \subseteq A$ and $a \in A$. We say that $a$ is a lower bound for $B$ iff
\[\forall x \in B(aRx)\].

20. Definition. Suppose that $R$ is a partial order on $A$. Suppose that $B \subseteq A$ and $a \in A$. We say that $a$ is an upper bound for $B$ iff
\[\forall x \in B(xRa)\].

21. Definition. Suppose that $R$ is a partial order on $A$, and $B \subseteq A$. Let $L$ be the set of all lower bounds for $B$. If $L$ has a largest element, then this largest element is called the greatest lower bound of $B$.

22. Definition. Suppose that $R$ is a partial order on $A$, and $B \subseteq A$. Let $U$ be the set of all upper bounds for $B$. If $U$ has a smallest element, then this smallest element is called the least upper bound of $B$.

23. Definition. Suppose that $A$ is a set and $\mathcal{F} \subseteq \mathcal{P}(A)$. We say that $\mathcal{F}$ is pairwise disjoint iff every pair of distinct elements of $\mathcal{F}$ are disjoint. We say that $\mathcal{F}$ is a partition of $A$ iff $\mathcal{F}$ is pairwise disjoint, $\bigcup \mathcal{F} = A$, and $\emptyset \notin \mathcal{F}$.

24. Definition. $R$ is an equivalence relation iff $R$ is reflexive, transitive, and symmetric.
25. Definition. Suppose that $R$ is an equivalence relation on $A$. Suppose that $x \in A$. The equivalence class of $x$ denoted $[x]$ is given by

$$[x] = \{y \in A | xRy\}.$$ 

We let $A/R$ (in words, $A$ modulo $R$) denote the set of equivalence classes.

26. Theorem. Suppose that $R$ is an equivalence relation on $A$. Suppose that $x, y \in A$. Then $xRy$ iff $[x] = [y]$.

27. Theorem. Suppose that $R$ is an equivalence relation on $A$. Then $A/R$ is a partition of $A$.

28. Theorem. Suppose that $A$ is a set, and $\mathcal{F}$ is a partition of $A$. Then there is an equivalence relation $R$ on $A$ such that $A/R = \mathcal{F}$.

29. Definition. We let $\mathbb{Z}$ denote the set of integers,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}.$$ 

30. Theorem. The sum of two integers is an integer. The product of two integers is an integer. The additive inverse of an integer is an integer.

31. Theorem. (Division Algorithm) If $a$ and $b$ are integers with $a > 0$, then there exist unique integers $q$ and $r$ such that $b = qa + r$ and $0 \leq r < a$.

32. Definition. Let $a$ and $b$ be integers. We say that $a$ divides $b$ iff there exists an integer $c$ such that $b = ac$.

33. Definition. Let $m \geq 2$ be an integer, and let $x$ and $y$ be integers. We say that $x$ is congruent to $y$ modulo $m$ iff $m$ divides $x - y$. We use the notation,

$$x \equiv y \pmod{m}.$$ 

34. Theorem. Let $m \geq 2$ be an integer, and let $C_m$ denote the set of ordered pairs $(x, y) \in (\mathbb{Z} \times \mathbb{Z})$ such that

$$x \equiv y \pmod{m}.$$
Then $C_m$ is an equivalence relation. Moreover, there are exactly $m$
distinct equivalence classes given by $[0], [1], \ldots, [m - 1]$.

35. Remark and Definition. Consider the case $m = 2$ in the previous
theorem. There are two distinct equivalence classes, $[0], [1]$. Integers
in $[0]$ are called even. Integers in $[1]$ are called odd.

36. Definition: Suppose that $F$ is a relation from $A$ to $B$. We say that $F$
is a function from $A$ to $B$ iff for every $a \in A$ there is a unique $b \in B$
such that $(a, b) \in F$. We use the notation $F : A \to B$ to indicate
that $F$ is a function from $A$ to $B$. Also, if $a \in A$, we let $F(a)$ denote
the unique $b \in B$ such that $(a, b) \in F$.

37. Theorem: Suppose that $f$ and $g$ are functions from $A$ to $B$. Then $f = g$ if and only if $\forall a \in A (f(a) = g(a))$.

38. Theorem: Suppose that $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ and for every $a \in A$ we have $(g \circ f)(a) = g(f(a))$.

39. Definition and Remark. Suppose that $f : A \to B$. We say that $f$
is one-to-one iff for all $a_1 \in A$ and $a_2 \in A$ if $f(a_1) = f(a_2)$ then
$a_1 = a_2$. We say that $f$ is onto iff for every $b \in B$ there exists $a \in A$
with $f(a) = b$. Note that $f$ is onto if and only if $B$ is the range of $f$.

40. Theorem: Suppose that $f : A \to B$ and $g : B \to C$. If $f$ and $g$ are
one-to-one, then $g \circ f$ is one-to-one. If $f$ and $g$ are onto, then $g \circ f$
is onto.

41. Remark and Theorem. Suppose that $f : A \to B$. Then $f$ is also a
relation from $A$ to $B$. So the inverse relation $f^{-1}$ is defined and is
a relation from $B$ to $A$. We have the following theorem: $f^{-1}$ is a
function from $B$ to $A$ if and only if $f$ is one-to-one and onto.

42. Theorem: Suppose that $f : A \to B$ and $g : B \to A$. Suppose also
that $g \circ f = i_A$ and $f \circ g = i_B$. Then $g = f^{-1}$.

43. Axiom. (Well-ordering principle). Every nonempty subset of $\mathbb{N}$ has a
smallest element.
44. Theorem. (Mathematical Induction). Suppose $j \in \mathbb{N}$. Suppose that $P(x)$ is a statement with a free variable. Suppose that

1. $P(j)$ and
2. For all $k \in \mathbb{N}$ with $k \geq j$ if $P(k)$ holds then $P(k + 1)$ also holds.

Then for all $n \in \mathbb{N}$ with $n \geq j$ we have $P(n)$.

45. Remark. Similar to Mathematical Induction, we sometimes use recursive definitions. We may define a function $f$ with domain $\mathbb{N}$ by defining $f(0)$ and defining $f(k + 1)$ in terms of $f(k)$. For example, if $x$ is a real number we may define $x^n$ by $x^0 = 1$ and $x^{(k+1)} = xx^k$.

46. Theorem. (Mathematical Induction, Strong Form). Suppose $j \in \mathbb{N}$. Suppose that $P(x)$ is a statement with a free variable. Suppose that

1. $P(j)$ and
2. For all $k \in \mathbb{N}$ with $k \geq j$ if $P(s)$ holds for all $s \in \mathbb{N}$ with $j \leq s \leq k$ then $P(k + 1)$ also holds.

Then for all $n \in \mathbb{N}$ with $n \geq j$ we have $P(n)$.