MHF 3202, Dr. Block, Chapter 1 Notes

1. If $P$ and $Q$ are statements, we can form three new statements:

$$P \land Q \text{ (and)}$$

$$P \lor Q \text{ (or)}$$

$$\neg P \text{ (negation)}$$

See the truth tables for these statements on page 15 of the text.

2. We can also form more complicated statements (also called formulas), such as

$$(P \land Q) \lor (\neg P \land \neg Q)$$

and construct truth tables for these formulas.

3. We say that statements $P$ and $Q$ are equivalent if and only if $P$ is true whenever $Q$ is true, and $P$ is false whenever $Q$ is false. See the laws on page 21 of the text.

4. If two formulas are equivalent, and one is substituted for other in some expression, the new expression obtained is equivalent to the original expression.

5. A formula which is always true is called a tautology. A formula which is always false is called a contradiction. See the laws on page 23 of the text.

6. We let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$, and $\mathbb{N}$ the set of natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$.

7. We use the notation $x \in A$ to denote that $x$ is an element of the set $A$. We use the notation $x \notin A$ to denote that $x$ is not an element of the set $A$.

8. Axiom: (Specification) If $A$ is a set and $P(x)$ is a statement about elements of $A$ there is a set which consists of all elements of $A$ such that $P(x)$ is true. This set is denoted by $\{x \in A \mid P(x)\}$ and is called the truth set of the statement $P(x)$. 
9. Suppose that $A$ and $B$ are sets. We say that $A$ is a subset of $B$ denoted $A \subseteq B$ if and only if every element of $A$ is also an element of $B$. We assume that two sets $A$ and $B$ are equal if and only if $A \subseteq B$ and $B \subseteq A$. Using this to prove that two sets are equal is called the method of double containment.

10. Axiom and Notation. Suppose that $A$ and $B$ are sets. We assume that there exist sets $A \cap B$, $A \cup B$, and $A \setminus B$ given by

- $x \in A \cap B$ if and only if $x \in A$ and $x \in B$,
- $x \in A \cup B$ if and only if $x \in A$ or $x \in B$,
- $x \in A \setminus B$ if and only if $x \in A$ and $x \notin B$.

These sets are called the intersection, union, and difference of the sets $A$ and $B$.

11. If $P$ and $Q$ are statements, we can also form the two new statements:

- $P \rightarrow Q$ (implies) (conditional statement)
- $P \leftrightarrow Q$ (is equivalent to) (biconditional statement).

See the truth tables for these statements on pages 45 and 52 of the text.

12. Note that $P \rightarrow Q$ is equivalent to $\neg P \lor Q$, and $\neg (P \rightarrow Q)$ is equivalent to $P \land \neg Q$.

13. Also, $P \lor Q$ is equivalent to $\neg P \rightarrow Q$. This equivalence is often used to prove a statement of the form $P \lor Q$.


- $P \rightarrow Q$ is equivalent to $\neg Q \rightarrow \neg P$.

15. If $C$ is a contradiction, then $(P \land \neg Q) \rightarrow C$ is equivalent to $P \rightarrow Q$.

16. Here are some ways to express the statement $P \rightarrow Q$:

- If $P$ then $Q$.
- $P$ implies $Q$.
- $P$ is a sufficient condition for $Q$. 
Q is a necessary condition for P.
P only if Q.

17. Here are some ways to express the statement $P \leftrightarrow Q$:

- $P$ if and only if $Q$.
- $Q$ is a necessary and sufficient condition for $P$.

18. The converse of a statement $P \rightarrow Q$ is the statement $Q \rightarrow P$. The converse of a true statement need not be true.

19. To prove a statement of the form $P \rightarrow Q$ with a direct proof:

- Begin the proof with "Suppose $P$"
- End the proof by proving $Q$.

20. To prove a statement of the form $P \rightarrow Q$ with a proof by contradiction: Begin the proof with "Suppose $P$".

- Next say: "Proceeding by contradiction, suppose $\neg Q$.
- End the proof by proving a contradiction.

21. To prove a statement of the form $P \leftrightarrow Q$ the most common way is to prove that both $P \rightarrow Q$ and $Q \rightarrow P$.

22. To prove a statement of the form $P \land Q$ the most common way is to prove $P$ and $Q$ separately.

23. Here are some ways to prove a statement of the form $P \lor Q$.

* Suppose $\neg P$ and prove $Q$.
* Suppose $\neg Q$ and prove $P$.
* Proceed by contradiction.
* Make cases and deal with each case separately. When you do this the cases must cover every possibility. Here is one example:

Case 1. $x \in A$.
Case 2. $x \notin A$. 
24. We introduce the universal quantifier \( \forall \) and the existential quantifier \( \exists \). A statement of the form \( \forall x \ (P(x)) \) means that for all \( x \) in the universe \( P(x) \) is true. A statement of the form \( \forall x \in B \ (P(x)) \) means that for all \( x \) in the set \( B \), \( P(x) \) is true. A statement of the form \( \exists x \ (P(x)) \) means that there exists \( x \) in the universe \( P(x) \) is true. A statement of the form \( \exists x \in B \ (P(x)) \) means that there exists \( x \) in the set \( B \) with \( P(x) \) true. In mathematics the words ”for some \( x \)” mean ”there exists \( x \)”.

25. Quantifier Negation laws:
- \( \neg \exists x \ (P(x)) \) is equivalent to \( \forall x \neg P(x) \).
- \( \neg \forall x \ (P(x)) \) is equivalent to \( \exists x \neg P(x) \).
- \( \neg \exists x \in A \ (P(x)) \) is equivalent to \( \forall x \in A \neg P(x) \).
- \( \neg \forall x \in A \ (P(x)) \) is equivalent to \( \exists x \in A \neg P(x) \).

26. The statements \( \forall x (x \in A \rightarrow P(x)) \) and \( \forall x \in A \ (P(x)) \) are equivalent. To prove a statement of this form, we begin the proof with ”Suppose \( x \in A \). Then we prove \( P(x) \).

27. For example, the logical form of the statement \( A \subseteq B \) is \( \forall x(x \in A \rightarrow x \in B) \) or \( \forall x \in A \ (x \in B) \). To prove that \( A \subseteq B \) we begin the proof with ”Suppose \( x \in A \)” and end the proof with ”\( x \in B \)”.

28. To prove a statement of the form \( \exists x \in A \ (P(x)) \):
- Begin the proof by exhibiting a particular \( x_0 \in A \).
- End the proof by proving that \( P(x_0) \) holds.

29. The notation \( \exists! x \in A \ (P(x)) \) means ”there exists a unique \( x \in A \) such that \( P(x) \) holds. To prove a statement of this form:
- Begin the proof of existence by exhibiting a particular \( x_0 \in A \).
- End the proof of existence by proving that \( P(x_0) \) holds.
- Prove uniqueness as follows:
  Suppose \( x \in A \) and \( P(x) \) holds. Then prove that \( x = x_0 \).

30. An alternate way to prove uniqueness is: Suppose that \( x_1 \in A \), \( x_2 \in A \) and both \( P(x_1) \) and \( P(x_2) \) hold. Then prove that \( x_1 = x_2 \).
31. Here are two rules of inference that we sometimes use.

(modus ponens) If $P$ and $P \rightarrow Q$ are both true, we can conclude that $Q$ is true.

(modus tollens) If $Q$ is false and $P \rightarrow Q$ is true, we can conclude that $P$ is false.

32. To prove a statement of the form $\neg P$ consider proof by contradiction. Another approach is to reexpress $\neg P$ as a positive statement.

33. Suppose we are given a statement or we are able to obtain a statement of the form $\exists x \in A$ such that $P(x)$ is true in a proof. To use this we may say ”we let $x_0$ denote an element of $A$ such that $P(x_0)$ is true”. Sometimes we say ”we may choose $x_0 \in A$ such that $P(x_0)$ is true”. But we can not assume anything additional about $x_0$. So we do not really have a choice to make. This is called existential instantiation.

34. Suppose we are given a statement or we are able to obtain a statement of the form $\forall x \in A, P(x)$ is true in a proof. To use this we must have or introduce a particular $a \in A$ to apply this statement to. Here we do sometimes really have a choice to make. This is called universal instantiation.

35. To prove a statement of the form $P \leftrightarrow Q$ the most common way is to prove that both $P \rightarrow Q$ and $Q \rightarrow P$. 