**MHF 3202, Dr. Block, Quiz 4 with answers, Spring 2020**

**Use the method of direct proof to prove each of the following statements.**

1. (2 points) If $x$ is an even integer, then $x^2$ is even.

   **Proof:** Suppose that $x$ is an even integer. Then for some integer $k$, we have $x = 2k$. It follows that $x^2 = 4k^2 = 2(2k^2)$. Also, $2k^2$ is an integer. Therefore, $x^2$ is even.

2. (2 points) Suppose that $a, b \in \mathbb{Z}$. If $a \mid b$, then $a^2 \mid b^2$.

   **Proof:** Suppose that $a, b \in \mathbb{Z}$, and $a \mid b$. Then for some integer $k$, we have $b = ak$. It follows that $b^2 = (a^2)(k^2)$. Also, $k^2$ is an integer. Therefore, $a^2 \mid b^2$.

3. (3 points) If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$.

   **Proof:** Suppose that $x \in \mathbb{R}$ and $0 < x < 4$. Since the square of any real number is greater than or equal to zero, we have $(x-2)^2 \geq 0$. By algebra, we obtain $x^2 - 4x + 4 \geq 0$. Adding $-4$ to each side of the inequality, we have $x^2 - 4x \geq -4$. Now, multiplying each side by $-1$ and factoring, we obtain

   $$x(4-x) \leq 4.$$

   Since $0 < x < 4$, using order properties of $\mathbb{R}$ we have $\frac{1}{x(4-x)} > 0$. So multipying each side of the displayed inequality by $\frac{1}{x(4-x)}$, we obtain $1 \leq \frac{4}{x(4-x)}$. Therefore, $\frac{4}{x(4-x)} \geq 1$.

4. (3 points) If $a$ is an integer and $a^2 \mid a$, then $a \in \{-1, 0, 1\}$.

   **Proof:** Suppose that $a$ is an integer and $a^2 \mid a$. Then for some integer $k$, we have $a = a^2k$. We have 2 cases.

   Case 1. $a = 0$. Then $a \in \{-1, 0, 1\}$.

   Case 2. $a \neq 0$. Then multiplying each side of the equality $a = a^2k$ by $\frac{1}{a}$, we obtain $1 = ak$. Since $k$ and $a$ are integers, either $k = a = 1$ or $k = a = -1$. Thus, $a \in \{-1, 0, 1\}$.

   Therefore, in all possible cases we have $a \in \{-1, 0, 1\}$.