MHF 3202, Dr. Block, Sample Exam #2 with solutions
There are six problems worth seven points each.

1. Let $a, b, c, d$ be real numbers such that $0 < a < b$ and $d > 0$. Prove that if $ac \geq bd$ then $c > d$.
   **Proof:** Suppose that $ac \geq bd$. Since $b > a$ and $d > 0$ we get $bd > ad$. Since $ac \geq bd$ it follows that $ac > ad$. Since $a > 0$ we have $a^{-1} > 0$, so $a^{-1}ac > a^{-1}ad$. Hence $c > d$.

2. Prove that for every real number $x$, if $x \neq 0$, then there is a unique real number $y$ such that for every real number $z$ we have $zy = z/x$.
   **Proof:** Suppose that $x$ is a real number, and $x \neq 0$.
   (Existence) Since $x \neq 0$ we can set $y_0 = 1/x$. Then for any $z \in \mathbb{R}$ we get $zy_0 = z/x$, as required.
   (Uniqueness) Suppose that $y \in \mathbb{R}$ satisfies $zy = z/x$ for every $z \in \mathbb{R}$. By considering $z = 1$ we get $y = 1y = 1/x$.
   So $y = 1/x$ is the unique real number with the specified property.

3. Prove that for every real number $x$ such that $x > 2$ there is a real number $y$ such that $y + \frac{1}{y} = x$.
   **Proof:** Suppose that $x$ is a real number with $x > 2$. Then $x^2 - 4 > 0$, so $y_0 = \frac{1}{2}(x + \sqrt{x^2 - 4})$ is a real number. We have
   \[ y_0 + \frac{1}{y_0} = \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2}{x + \sqrt{x^2 - 4}} = \frac{x + \sqrt{x^2 - 4}}{2} + \frac{x - \sqrt{x^2 - 4}}{2} = x. \]
   Therefore $y = y_0$ is a solution to $y + \frac{1}{y} = x$.

4. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets such that every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$. Prove that $\bigcup \mathcal{F} \subseteq \bigcap \mathcal{G}$.
   **Proof:** Suppose that $x \in \bigcup \mathcal{F}$. Then there is $A \in \mathcal{F}$ such that $x \in A$. Let $B \in \mathcal{G}$ be arbitrary. Then $A \subseteq B$ by assumption. Hence $x \in B$. Since this holds for every $B \in \mathcal{G}$ we get $x \in \bigcap \mathcal{G}$. We conclude that $\bigcup \mathcal{F} \subseteq \bigcap \mathcal{G}$. 
5. Prove that $\forall x \in \mathbb{R}((\exists y \in \mathbb{R} (x + y = xy)) \leftrightarrow x \neq 1)$.

**Proof:** Suppose that $x \in \mathbb{R}$.

First, suppose that $x \neq 1$. Set $y_0 = x/(x - 1)$ and observe that

$$xy_0 = \frac{x^2}{x - 1} = x + \frac{x}{x - 1} = x + y_0.$$ 

Therefore if $x \neq 1$ then there does exist $y \in \mathbb{R}$ such that $x + y = xy$. On the other hand, if $x = 1$ then the equation $x + y = xy$ reduces to $1 + y = y$, which has no solution in $y$. Therefore there exists $y \in \mathbb{R}$ such that $x + y = xy$ if and only if $x \neq 1$.

6. Let $A$, $B$, $C$ be sets such that $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Prove that $A \subseteq B$.

**Proof:** Suppose that $x \in A$. Since $A \cup C \subseteq B \cup C$, we get $x \in B \cup C$. Therefore either $x \in B$ or $x \in C$. If $x \in B$ then we have the conclusion we want. If $x \in C$ then since $x \in A$ we get $x \in A \cap C$. Since $A \cap C \subseteq B \cap C$, we see that $x \in B \cap C$. It follows that $x \in B$ in this case too. Since $x \in A$ implies $x \in B$ in all cases, we have $A \subseteq B$. 