MHF 3202, Dr. Block, Sample Exam #2, with answers
There are 7 problems worth a total of 40 points.

1. (4 points) Complete the following definition: Let $R$ be a relation on $A$. Then $R$ is antisymmetric iff

   **Answer:**
   $\forall x \in A \forall y \in A, ((xRy \land yRx) \rightarrow x = y)$.

2. (4 points) Complete the following definition: Suppose that $R$ is a partial order on $A$. Suppose that $B \subseteq A$ and $b \in B$. We say that $b$ is a minimal (or $R$-minimal) element of $B$ iff

   **Answer:**
   $\neg \exists x \in B (xRb \land x \neq b)$.

3. (4 points) Complete the following definition: Let $R$ be a relation on $A$. Then $R$ is an equivalence relation on $A$ iff

   **Answer:**
   $R$ is reflexive, transitive, and symmetric.

4. (2 points) Determine if the statement is true or false: If $R_1$ and $R_2$ are partial orders on a set $A$, then $R_1 \cup R_2$ is a partial order on $A$.

   **Answer:** False.

5. (2 points) Determine if the statement is true or false: Let $A = \mathcal{P}(\{1, 2, 3, 4\})$ and let $R$ be the partial order on $A$ defined by

   $$R = \{(X,Y) \in A \times A \mid X \subseteq Y\}.$$

   If $B = \{\{1, 2, 3\}, \{2, 3, 4\}\}$. Then $\{2, 3\}$ is the greatest lower bound of $B$.

   **Answer:** True.
6. (8 points) Let $U$ be any set. Prove that there is a unique $A \in \mathcal{P}(U)$ such that for every $B \in \mathcal{P}(U)$, $A \cap B = B$.

**Answer:**

First, we prove existence. Set $A = U$. Suppose that $B \in \mathcal{P}(U)$. Then $B \subseteq U$. It follows that

$$A \cap B = U \cap B = B.$$ 

Next, we prove uniqueness. Suppose that $A \in \mathcal{P}(U)$ and for every $B \in \mathcal{P}(U)$, $A \cap B = B$. Since $U \in \mathcal{P}(U)$, we have $A \cap U = U$. On the other hand, as $A \subseteq U$, we have that $A \cap U = A$. It follows that $A = U$.

7. (8 points) Suppose $R$ is a relation from $A$ to $B$ and $S$ and $T$ are relations from $B$ to $C$. Prove that

$$(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R.$$ 

**Answer:**

Suppose that $(x, z) \in (S \circ R) \setminus (T \circ R)$. Then $(x, z) \in (S \circ R)$, and $(x, z) \notin (T \circ R)$. Since $(x, z) \in (S \circ R)$, we can choose $y \in B$ such that $(x, y) \in R$ and $(y, z) \in S$.

We claim that $(y, z) \notin T$. We prove this claim by contradiction. Suppose that $(y, z) \in T$. Since $(x, y) \in R$ we have $(x, z) \in (T \circ R)$. This is a contradiction. This proves the claim.

We now have $(x, y) \in R$ and $(y, z) \in S \setminus T$. It follows that $(x, z) \in (S \setminus T) \circ R$. Since $(x, z)$ was arbitrary, we have

$$(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R.$$
8. (8 points) Prove that for every integer $n$ either $n^2 \equiv 0 \pmod{4}$ or $n^2 \equiv 1 \pmod{4}$.

**Answer:** Suppose that $n$ is an integer. Then either $n$ is even or $n$ is odd. We have 2 cases.

Case 1. $n$ is even. Then for some integer $k$, we have $n = 2k$. It follows that $n^2 - 0 = 4k^2$. Since $k^2$ is an integer, we have $n^2 \equiv 0 \pmod{4}$.

Case 2. $n$ is odd. Then for some integer $k$, we have $n = 2k + 1$. It follows that $n^2 = 4k^2 + 4k + 1$, and so $n^2 - 1 = 4(k^2 + k)$. Since $k^2 + k$ is an integer, we have $n^2 \equiv 1 \pmod{4}$. 