301. Definition. Let $m$ be a positive integer, and let $X$ be a set. An $m$-tuple of elements of $X$ is a function

$$x : \{1, \ldots, m\} \to X.$$  

We sometimes use $x_i$ instead of $x(i)$, and sometimes denote $x$ by $(x_1, \ldots, x_m)$.

302. Definition. Let $m$ be a positive integer, let $\{A_1, \ldots, A_m\}$ be an indexed collection of sets, and let $X = A_1 \cup \cdots \cup A_m$. We define the Cartesian product of the indexed family to be the set of all $m$-tuples $(x_1, \ldots, x_m)$ such that $x_i \in A_i$ for each $i$. We denote the Cartesian product by $\prod_{i=1}^m A_i$ or $A_1 \times \cdots \times A_m$. If all of the sets $A_i$ are the same, we denote the Cartesian product by $X^m$.

303. Definition. Let $X$ be a set. An $\omega$-tuple of elements of $X$ is a function $x : \mathbb{Z}_+ \to X$. We also call such a function a sequence or an infinite sequence of elements of $X$. We sometimes use $x_i$ instead of $x(i)$, and sometimes denote $x$ by $(x_1, x_2, \ldots)$ or $(x_i)_{i \in \mathbb{Z}_+}$.

304. Definition. Let $\{A_1, A_2, \ldots\}$ be an indexed collection of sets, and let $X = (A_1 \cup A_2 \cup \ldots)$. We define the Cartesian product of the indexed family to be the set of all $\omega$-tuples $(x_1, x_2, \ldots)$ such that $x_i \in A_i$ for each $i$. We denote the Cartesian product by $\prod_{i \in \mathbb{Z}_+} A_i$ or $(A_1 \times A_2 \times \ldots)$. In the special case where all of the sets $A_i$ are the same, we denote the Cartesian product by $X^\omega$. See chapter 1, section 5 of the text for details.

305. Definition. A set $S$ is countably infinite if and only if there is a bijection $f : S \to \mathbb{Z}_+$. A set is countable if and only if it is either finite or countably infinite. A set is uncountable if it is not countable.

306. Theorem. Let $S$ be a nonempty set. The following are equivalent.

1. $S$ is countable.
2. There is a surjective function $f : \mathbb{Z}_+ :\to S$.
3. There is an injective function $g : S \to \mathbb{Z}_+$.

307. Theorem. An infinite subset of $\mathbb{Z}_+$ is countably infinite.

308. Corollary. Any subset of a countable set is countable.
309. Theorem. A countable union of countable sets is countable. A finite product of countable sets is countable.

310. Theorem. Let $X = \{1, 2\}$. Then $X^\omega$ is uncountable.

311. Theorem. Let $S$ be a set, and let $P(S)$ denote the set of subsets of $S$. There is no surjective map $f : S \to P(S)$.

312. Corollary. Let $S$ be a set, and let $P(S)$ denote the set of subsets of $S$. There is no injective map $f : P(S) \to S$.

313. Proposition. Let $m$ be a positive integer, and let $\{X_1, \ldots, X_m\}$ be an indexed collection of topological spaces. Let $E = \prod_{i=1}^{m} X_i$. Let $\mathcal{B}$ denote the collection of all subsets $B$ of $E$ such that $B = \prod_{i=1}^{m} W_i$, where $W_i$ is an open subset of $E_i$ for each $i$. Then $\mathcal{B}$ is a basis for a topology on $E$.

314. Definition. The topology generated by the basis in the previous Proposition is called the product topology. When we consider a finite product of topological spaces, unless otherwise specified we assume that the topology on the product is the product topology.

315. Proposition. Let $m \geq 3$ be a positive integer, and let $\{X_1, \ldots, X_m\}$ be an indexed collection of topological spaces. Then $\prod_{i=1}^{m} X_i$ is homeomorphic to $(\prod_{i=1}^{m-1} X_i) \times X_m$.

316. Corollary. A finite product of connected spaces is connected.


318. Definitions.
(a) A topological space $X$ is said to be second countable if and only if there is a countable basis for the topology.
(b) A topological space $X$ is said to be Lindelof if and only if every open cover of $X$ has a countable subcover.
(c) A subset $B$ of a topological space $X$ is said to be dense in $X$ if and only if $\overline{B} = X$.
(d) A topological space $X$ is said to be separable, if and only if there exist a countable, dense subset of $X$.

319. Proposition. Let $X$ be a second countable topological space. Then $X$ is separable.

320. Proposition. Let $X$ be a second countable topological space. Then $X$ is Lindelof.
321. Example. Give an example of a topological space which is Lindelof, but not separable.

322. Example. Give an example of a topological space which is separable, but not Lindelof.

323. Definitions.

(a) Let $I$ be any set, and let $\{X_i : i \in I\}$ be an indexed family of sets. The Cartesian Product of these sets is defined to be

$$\prod_{i \in I} X_i = \{x : I \to (\bigcup_{i \in I} X_i) : \forall i \in I, x(i) \in X_i\}$$

We sometimes denote an element of the Cartesian Product by $(x_i)_{i \in I}$.

(b) Let $a \in I$. The projection

$$p_a : \prod_{i \in I} X_i \to X_a$$

is defined by $p_a(x) = x(a)$.

324. Axiom of Choice. The Cartesian Product of a non-empty family of non-empty sets is non-empty.

325. Definition. Let $I$ be any set, and let $\{(X_i, T_i) : i \in I\}$ be an indexed family of topological spaces. Let

$$X = \prod_{i \in I} X_i.$$ 

Let $S$ be the collection of subsets of $X$ of the form

$$W = \prod_{i \in I} W_i$$

where $W_i \in T_i$ for each $i \in I$ and $W_i = X_i$ for all except possible one $i \in I$. Observe that $S$ is a collection of subsets of $X$ whose union is $X$. Let $\mathcal{T}$ be the topology generated by the subbasis $S$. Then $\mathcal{T}$ is called the product topology on $X$. Whenever we talk about the product of topological spaces, we will assume that we are using the product topology.

326. Theorem. The product of Hausdorff spaces is a Hausdorff space.

Note: Here the index set $I$ is arbitrary.
327. Proposition. Let $X$ and $Y$ be topological spaces, and let $f : X \to Y$. Suppose that $S$ is a subbasis which generates the topology on $Y$. Then $f$ is continuous if and only if for each $W \in S$, the inverse image $f^{-1}(W)$ is an open subset of $X$.

328. Theorem. Suppose that $X = \prod_{i \in I} X_i$.

Suppose that $f : Y \to X$. Then $f$ is continuous if and only if each composition $p_a \circ f$ where $a \in I$ is continuous.

329. Proposition. Suppose that $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ are indexed families of topological spaces. Suppose that $\alpha : I \to J$ is a bijection and for all $i \in I$, we have $X_i = Y_{\alpha(i)}$. Then $\prod_{i \in I} X_i$ is homeomorphic to $\prod_{j \in J} Y_j$.

330. Proposition. Suppose that $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are indexed families of topological spaces. Suppose that for each $i \in I$, the spaces $X_i$ and $Y_i$ are homeomorphic. Then $\prod_{i \in I} X_i$ and $\prod_{i \in I} Y_i$ are homeomorphic.

331. Theorem. The countable product of metrizable spaces is a metrizable space.

332. Problem. Let $A_1 = [0, 1]$, $A_2 = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and define $A_n$ inductively by deleting the open middle third interval of each closed interval from the previous stage. Let $C = \cap_{n=1}^{\infty} A_n$. Then $C$ is called the Cantor set. Prove that

1. $C$ is closed (as a subset of $\mathbb{R}$).
2. No open interval is a subset of $C$.
3. For each $x \in C$ we have $x \in \overline{C - \{x\}}$, so $x$ is a limit point of $C$.

333. Problem. Let $X$ denote the set $\{1, 2\}$ with the discrete topology. Prove that $X^{\omega}$ is homeomorphic to the Cantor set. In particular, note that $X^{\omega}$ does not have the discrete topology.

334. Lemma. Let $I$ be an index set, and suppose that $I$ is the union of two disjoint subsets $J$ and $K$. Let $\{(X_i, T_i) : i \in I\}$ be an indexed family of topological spaces. Then

$$\prod_{i \in I} X_i$$

is homeomorphic to

$$\prod_{i \in J} X_i \times \prod_{i \in K} X_i$$

335. Problem. Prove the product of the Cantor set with itself is homeomorphic to the Cantor set.
336. Theorem. The product of connected spaces is connected.

Hint: Recall that the finite product of connected spaces is connected. In the general case, consider the component of a point.

337. Definitions. A partial order on a set $X$ is a reflexive, antisymmetric, transitive relation on $X$. A partially ordered set is an ordered pair $(X, \leq)$ such that $X$ is a set, and $\leq$ is a partial order on $X$.

Let $(X, \leq)$ be a partially ordered set, and let $A \subseteq X$.

(a) An element $w \in X$ is called a maximal element of $X$ if and only if for every $x \in X$, $w \leq x$ implies that $w = x$.

(b) An element $u \in X$ is called an upper bound of $A$ if and only if $x \leq u$ for every $x \in A$.

(c) $A$ is said to be a chain in $X$ if and only if for every pair $x, y \in A$ either $x \leq y$ or $y \leq x$.

338. Theorem. (Zorn’s Lemma) If $(X, \leq)$ is a partially ordered set such that every chain in $X$ has an upper bound, then there is a maximal element of $X$.

Note: This can be proved using the Axiom of Choice. We will not prove this here.

339. Definitions. Suppose that $S$ is a subbasis for the topology on $X$. We say that a subset $W$ of $X$ is a subbasic open set if and only if $W \in S$. We say that a subset $W$ of $X$ is a subbasic closed set if and only if $(X - W) \in S$.

340. Theorem. Suppose that $S$ is a subbasis for the topology on $X$. Then $X$ is compact if and only if the following holds:

$\text{e. If } \mathcal{D} = \{D_i : i \in I\}$ is a collection of subbasic closed subsets of $X$, and $\mathcal{D}$ has the finite intersection property, then

$$\bigcap_{i \in I} D_i \neq \emptyset.$$  

Hint: Use Zorn’s Lemma. The proof is not easy.

341. Theorem. The product of compact spaces is compact.

342. Problem. Prove that if a metrizable space $X$ is separable, then $X$ is second countable.

343. Problem. Prove that if a metrizable space $X$ is Lindelof, then $X$ is second countable.
344. Problem. Let \( I \) be any set, and let \( \{ X_i : i \in I \} \) be an indexed family of topological spaces. Let \( X = \prod_{i \in I} X_i \). Suppose that \( \{ A_i : i \in I \} \) is an indexed family of sets with \( A_i \subset X_i \) for each \( i \in I \). Prove that \( \prod_{i \in I} A_i \) is dense in \( X \) if and only if \( A_i \) is dense in \( X_i \) for each \( i \in I \).

345. Problem. Prove that if a metrizable space \( X \) is connected and contains more than one point, then \( X \) is uncountable.

Hint: You may use the fact that any closed interval \([a,b]\), (where \( a \) and \( b \) are real numbers with \( a < b \)) is uncountable.

346. Problem. Find all components of \( \mathbb{R}_\ell \). (See Example 74.) Find all continuous maps \( f : \mathbb{R} \to \mathbb{R}_\ell \).

347. Definition and Remark. Let \( n \) be a positive integer. We define two metrics on \( \mathbb{R}^n \). Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \). We define the euclidean metric by
\[
d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.
\]
We define the square metric by
\[
d(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.
\]
It can be verified that these are indeed metrics.

348. Theorem. The topologies on \( \mathbb{R}^n \) induced by the euclidean metric and the square metric are the same as the product topology on \( \mathbb{R}^n \).

349. Theorem. Let \( A \) be a subset of \( \mathbb{R}^n \). Consider either the euclidean metric or the square metric on \( \mathbb{R}^n \). Then \( A \) is compact if and only if \( A \) is closed and bounded.

350. Definition and Remark. We define a new metric on the set \( \mathbb{R}^\omega \) as follows. If \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \), we set
\[
d(x,y) = \sup\{d_1(x_1, y_1), d_2(x_2, y_2), \ldots\}
\]
where \( d_i(x_i, y_i) = \min\{|x_i - y_i|, 1\} \) for each positive integer \( i \). This metric is called the uniform metric, and the topology induced from this metric is called the uniform topology.

351. Theorem. The uniform topology on \( \mathbb{R}^\omega \) is finer than the product topology on \( \mathbb{R}^\omega \).

352. Problem. Is the uniform topology on \( \mathbb{R}^\omega \) strictly finer than the product topology on \( \mathbb{R}^\omega \)?

353. Definition. Let \( (X, d) \) be a metric space, and let \( (x_n) \) be a sequence of points in \( X \). We say that \( (x_n) \) is a Cauchy sequence if and only if for every \( \epsilon > 0 \) there is a positive integer \( k \) such that if \( i, j \geq k \), then \( d(x_i, x_j) < \epsilon \).
354. Proposition. Let \((X,d)\) be a metric space, and let \((x_n)\) be a sequence of points in \(X\). If \((x_n)\) converges to a point in \(X\), then \((x_n)\) is a Cauchy sequence.

355. Problem. Is \([0,1]^\omega\) with the uniform topology compact?

356. Definition. A metric space \((X,d)\) is said to be complete if and only if every Cauchy sequence in \(X\) converges to a point in \(X\).

357. Proposition. Let \((X,d)\) be a metric space, and let \((x_n)\) be a sequence of points in \(X\). If \((x_n)\) is a Cauchy sequence, then the set \(\{x_n : n \in \mathbb{Z}_+\}\) is bounded.

358. Proposition. Let \((X,d)\) be a metric space, and let \((x_n)\) be a Cauchy sequence of points in \(X\). If some subsequence \((x_{n_k})\) of \((x_n)\) converges to a point \(x \in X\), then the sequence \((x_n)\) converges to \(x\).

359. Theorem. A compact metric space is complete.

360. Proposition. A closed subset of a complete metric space is complete.

361. Theorem. \(\mathbb{R}^n\) with either the euclidean metric or the square metric is complete.

362. Proposition. Let \((X,d)\) be a complete metric space. Suppose that \((A_n)\) is a nested sequence of nonempty, closed subsets of \(X\) such that \(\lim_{n \to \infty} d(A_n) = 0\). Then \(\bigcap_{n=1}^{\infty} A_n\) is nonempty and consists of a single point.

363. Theorem. (Baire Category Theorem) Let \((X,d)\) be a complete metric space. If \(\{V_1, V_2, \ldots\}\) is a countable collection of open dense subsets of \(X\), then \(\bigcap_{n=1}^{\infty} V_n\) is dense in \(X\).

364. Definition. Let \(X\) be a topological space, and let \(x \in X\). We say that \(x\) is an isolated point of \(X\) if and only if the set \(\{x\}\) is open.

365. Problem. Let \((X,d)\) be a nonempty complete metric space with no isolated points. Prove that \(X\) is uncountable.

366. Definition. Let \((X,d)\) be a metric space, and let \(f : X \to X\). We say that \(f\) is a contraction mapping if and only if there is a real number \(c\) with \(0 < c < 1\) such that for every pair of points \(x, y \in X\) we have

\[
d(f(x), f(y)) \leq c \cdot d(x, y).
\]

367. Definition. Let \(X\) be a topological space, and let \(f : X \to X\). Let \(x \in X\). We say that \(x\) is a fixed point of \(f\) if and only if \(f(x) = x\).
368. Proposition. Let \((X, d)\) be a metric space, and let \(f : X \to X\) be continuous. Let \((y_n)\) be a sequence of points in \(X\) such that \(y_0\) is a point of \(X\), and for each \(n \geq 1\), we have \(y_n = f(y_{n-1})\). If the sequence \((y_n)\) converges to a point \(x \in X\), then \(x\) is a fixed point of \(f\).

369. Theorem. (Contraction Mapping Theorem) Let \((X, d)\) be a nonempty complete metric space, and let \(f : X \to X\). If \(f\) is a contraction mapping, then \(f\) has a unique fixed point.

370. Problem. Is \(\mathbb{R}\) with the uniform topology connected?

371. Problem. Consider \(X = \mathbb{R}\) with the product topology. For each positive integer \(k\), let \(D_k\) denote the set of points \(x = (x_1, x_2, \ldots)\) such that \(x_i = 0\) for all \(i \geq k\). Let \(D = \bigcup_{k=1}^{\infty} D_k\). Is \(D\) a closed subset of \(X\)? Is \(D\) dense in \(X\)?

372. Definition. (called separation axioms) Let \(X\) be a topological space.

\(X\) is called \(T_0\) if and only if for every pair of distinct points \(x, y \in X\) there is an open set which contains \(x\) but not \(y\), or there is an open set which contains \(y\) but not \(x\).

\(X\) is called \(T_1\) if and only if for every pair of distinct points \(x, y \in X\), there is an open set which contains \(x\) but not \(y\), and there is an open set which contains \(y\) but not \(x\).

\(X\) is called \(T_2\) or Hausdorff if and only if for every pair of distinct points \(x, y \in X\), there are disjoint open sets \(U\) and \(V\) with \(x \in U\) and \(y \in V\).

\(X\) is called \(T_3\) or regular if and only if it is \(T_1\) and for every \(x \in X\) and every closed subset \(K\) of \(X\) with \(x \notin K\) there are disjoint open sets \(U\) and \(V\) with \(x \in U\) and \(K \subset V\).

\(X\) is called \(T_4\) or normal if and only if it is \(T_1\) and for every pair of disjoint closed subsets \(K\) and \(D\) of \(X\) there are disjoint open sets \(U\) and \(V\) with \(K \subset U\) and \(D \subset V\).

373. Proposition. Let \(X\) be a topological space.

a. If \(X\) is \(T_1\), then \(X\) is \(T_0\).

b. If \(X\) is \(T_2\), then \(X\) is \(T_1\).

c. If \(X\) is \(T_3\), then \(X\) is \(T_2\).

d. If \(X\) is \(T_4\), then \(X\) is \(T_3\).

e. If \(X\) is metrizable, then \(X\) is \(T_4\).

374. Theorem. A compact Hausdorff space is normal.