TOPOLOGY, DR. BLOCK, FALL 2015, NOTES, PART 4

401. Definition. Let X and Y be topological spaces, and let $f : X \to Y$. We say f is an open map if and only if for every open subset V of X, $f(V)$ is open in Y. We say f is a closed map if and only if for every closed subset V of X, $f(V)$ is closed in Y.

402. Problem. Consider the projection $p_1 : \mathbb{R}^2 \to \mathbb{R}$. Is p_1 an open map? Is p_1 a closed map?

403. Definition. Let S^1 denote the set of $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $x^2 + y^2 = 1$. This space is called the circle.

404. Problem. Define a function $p : \mathbb{R} \to S^1$ by $p(t) = (\cos(2\pi t), \sin(2\pi t))$. Is p continuous? Is p an open map? Is p a closed map?

Hint: You may use the fact that the the sine and cosine functions are continuous, and other known facts about these functions.

405. Definition. Let X and Y be topological spaces, and let $p : X \to Y$ be surjective. We say p is a quotient map if and only if the following condition holds: A subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

406. Proposition. Let X and Y be topological spaces, and let $p : X \to Y$ be a surjective map. The function p is a quotient map if and only if the following condition holds: A subset K of Y is closed in Y if and only if $p^{-1}(K)$ is closed in X.

407. Definition. Let X and Y be topological spaces, and let $p : X \to Y$ be surjective. We say a subset A of X is saturated (with respect to p) if and only if $A = p^{-1}(B)$ for some subset B of Y.

408. Proposition. Let X and Y be topological spaces, and let $p : X \to Y$ be a continuous, surjective map. The function p is a quotient map if and only if the following condition holds: If A is a saturated open subset of X, then $p(A)$ is an open subset of Y.

409. Proposition. Let X and Y be topological spaces, and let $p : X \to Y$ be a continuous, surjective map. If p is an open map, then p is a quotient map.

410. Proposition. Let X and Y be topological spaces, and let $p : X \to Y$ be a continuous, surjective map. If p is a closed map, then p is a quotient map.

411. Proposition and Definition. Let X be a topological space, let S be a set, and let $p : X \to S$ be surjective. There exists a unique topology on S such that p is a quotient map. This topology is called the quotient topology induced by p.

412. Definition. Let X be a topological space, and let X^* be a partition of X (a collection of non-empty, pairwise disjoint, subsets of X, whose union is X). Let $p : X \to X^*$ be the surjective function which assigns to each point of X the element of X^* containing it. The space X^* with the quotient topology induced by p is called a quotient space of X.

413. Definition and Remark. Let X be a topological space. Let $\sim$ be an equivalence relation on X, and for each $x \in X$, let $[x]$ denote the equivalence class of x. Let $X/\sim$ denote the set of equivalence classes. The function $p : X \to X/\sim$ defined by $p(x) = [x]$ is called the projection. The set $X/\sim$ with the quotient topology induced by the function p is called the quotient space of X by $\sim$.

Recall that any equivalence relation on X corresponds to a partition of X and vice versa. Thus the notations X^* and $X/\sim$ are notations for the same topological space. So we will only use the notation $X/\sim$ for the quotient space obtained from either a partition of X or an equivalence relation on X.

414. Theorem. Let $p : X \to Y$ be a quotient map. Let Z be a topological space, and let $g : X \to Z$ be a function that is constant on each set $p^{-1}(\{y\})$ for $y \in Y$. Then g induces a function $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous. Also, f is a quotient map if and only if g is a quotient map.

415. Theorem. Let $g : X \to Z$ be a continuous surjective map. Let $X/\sim$ be the quotient space obtained from the partition of X which consists of all sets of the form $g^{-1}(\{z\})$ where $z \in Z$. Let $p : X \to X/\sim$ be defined by $p(x) = g^{-1}(\{g(x)\})$. (Observe that p is the projection as in 413 above.)

(i) The map g induces a bijective continuous map $f : X/\sim \to Z$ such that $f \circ p = g$. Moreover, f is a homeomorphism if and only if g is a quotient map.

(ii) If Z is Hausdorff, so is $X/\sim$.

416. Problem. Define an equivalence relation $\sim$ on $X = \mathbb{R}$ by $x \sim y$ if and only if $x - y$ is an integer. Prove that $X/\sim$ is homeomorphic to the circle S^1.

417. Problem. Define an equivalence relation $\sim$ on $X = [0,1]$ by $x \sim x$ for all x, $0 \sim 1$, and $1 \sim 0$. Prove that $X/\sim$ is homeomorphic to the circle S^1.

418. Problem. Define an equivalence relation $\sim$ on $X = \mathbb{R} \times \mathbb{R}$ by $(x, y) \sim (v, w)$ if and only if both $x - v$ and $y - w$ are integers. Prove that $X/\sim$ is homeomorphic to the torus $S^1 \times S^1$.

419. Problem. Define an equivalence relation $\sim$ on $X = [0,1] \times [0,1]$ by $(x, y) \sim (v, w)$ if and only if both $x - v$ and $y - w$ are integers. Prove that $X/\sim$ is homeomorphic to the torus $S^1 \times S^1$.

420. Problem. Let $p : X \to Y$ be a quotient map. Suppose that Y is connected, and for each $y \in Y$, the set $p^{-1}(\{y\})$ is connected. Prove that X is connected.

421. Theorem. $p : X \to Y$ be a quotient map. Let K be a subset of Y which is either open or closed. Let $g : p^{-1}(K) \to K$ denote the restriction of p to $p^{-1}(K)$. Then g is a quotient map.

422. Problem. Define an equivalence relation $\sim$ on a topological space X by $x \sim y$ if and only if x and y lie in the same component of X. Prove that each component of $X/\sim$ consists of a single point.

423. Problem. Let $p_1 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ denote the projection onto the first coordinate. Let

$$A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0\} \cup \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 0\}.$$

Let $g : A \to \mathbb{R}$ denote the restriction of p_1 to A. Is g continuous? Is g an open map? Is g a closed map? Is g a quotient map?

424. Problem. Give an example of an equivalence relation on $\mathbb{R}$ such that the corresponding quotient space is not Hausdorff.