A NEW UPPER BOUND FOR 1324-AVOIDING PERMUTATIONS

MIKLÓS BÓNA

Abstract. We prove that the number of 1324-avoiding permutations of length n is less than $(7+4\sqrt{3})^n$. The novelty of our method is that we injectively encode such permutations by a pair of words of length n over a finite alphabet that avoid a given factor.

1. Introduction

1.1. **Definitions and Open Questions.** The theory of pattern avoiding permutations has seen tremendous progress during the last two decades. The key definition is the following. Let $k \leq n$, let $p = p_1 p_2 \cdots p_n$ be a permutation of length n, and let $q = q_1 q_2 \cdots q_k$ be a permutation of length k. We say that p avoids q if there are no k indices $i_1 < i_2 < \cdots < i_k$ so that for all a and b, the inequality $p_{i_a} < p_{i_b}$ holds if and only if the inequality $q_a < q_b$ holds. For instance, $p = 2537164$ avoids $q = 1234$ because p does not contain an increasing subsequence of length four. See [3] for an overview of the main results on pattern avoiding permutations.

The shortest pattern for which even some of the most basic questions are open is $q = 1324$, a pattern that has been studied for at least 17 years. For instance, there is no known exact formula for the number $S_n(1324)$ of permutations of length n (or, in what follows, n-permutations) avoiding 1324. Even the value of $L(1324) = \lim_{n\to\infty} \sqrt[n]{S_n(1324)}$ is unknown, though the limit is known to exist. Indeed, a spectacular result of Adam Marcus and Gábor Tardos [7] shows that for all patterns q, there exists a constant c_q so that $S_n(q) \leq c_q^n$ for all n, and a short argument [2] then shows that this implies the existence of $L(q) = \lim_{n\to\infty} \sqrt[n]{S_n(q)}$. It is also known that no pattern of length four is easier to avoid than the pattern 1324, that is, for any pattern q of length 4, the inequality $S_n(q) \leq S_n(1324)$ holds. The inequality is sharp unless $q = 4231$. See Chapter 4 of [3] for a treatment of the series of results leading to these inequalities.

The best known upper bound for the numbers $S_n(1324)$ was given in 2011 by Claesson, Jelinek and Steingrímsson [5] who proved that for all positive integers n, the inequality $S_n(1324) < 16^n$ holds. The best known lower bound, $S_n(1324) \geq 9.47^n$, was given by five authors in [1] in 2005.

In this paper, we prove the inequality $S_n(1324) < (7+4\sqrt{3})^n$. The proof introduces a refined version of a decomposition of 1324-avoiding permutations given in [5], encodes such permutations by two words over a 4-element alphabet, and then enumerates those words. As far as we know, this is the first time that the combinatorics of words is used to find a good upper bound for the number of permutations avoiding a single pattern.

1.2. **Preliminaries.** In this section, we present a few simple facts that are well-known among researchers working in the area that will be necessary in order to understand some of our proofs in the subsequent sections. Readers familiar with the area may skip this section. Proofs that are not given here can be found in [3].

Theorem 1.1. *Let q be any pattern of length three. Then $S_n(q) = C_n = \binom{2n}{n}/(n+1)$, the nth Catalan number. In particular, $S_n(q) < 4^n$.*

An entry of a permutation is called a *left-to-right* minimum if it is smaller than all entries on its left. *Right-to-left* maxima are defined analogously. For instance, in $p = 351624$, the left-to-right minima are 3 and 1, while the right-to-left maxima are 6 and 4. The following proposition was first proved by Rodica Simion and Frank Schmidt in [8].

Proposition 1.2. *A 132-avoiding permutation is completely determined by the set of its left-to-right minima, and the set of indices that belong to entries that are left-to-right minima.*

Proof. By definition, left-to-right minima are always in decreasing order. Furthermore, once the set and position of the left-to-right minima are given, the order of elements that are not left-to-right minima is uniquely determined. To see this, fill the positions that belong to entries that are not left-to-right minima one by one, going left to right. In each step, the smallest remaining entry that is larger than the closest left-to-right minimum m on the left of the position at hand must be placed. If we do not follow this procedure and place the entry y instead of the smaller entry x, then the 132-pattern myx is formed. □

Example 1.3. *In order to find the unique 132-avoiding permutation of length 6 whose left-to-right minima are the entries 1, 3, and 4, and that has left-to-right minima in the first, second and fifth positions, write the left-to-right minima in the specified positions in decreasing order, to get $43 * *1*$, where the $*$ denote positions that are still empty. Then fill the empty slots with the remaining entries, always placing the smallest entry that is larger than the closest left-to-right minimum on the left. In this case, that means first placing 5, then 6, then 2, to get* 435612.

In an analogous way, each 213-avoiding permutation is determined by the set of its right-to-left maxima, and the set of indices that belong to right-to-left maxima. This is easy to see if we observe that the permutation $p = p_1p_2 \cdots p_n$ is 213-avoiding if and only if its reverse complement, that is, the permutation $(n+1-p_n)\ (n+1-p_{n-1})\ \cdots (n+1-p_1)$ is 132-avoiding.

In preparation to proving our main results, we announce the facts discussed in Proposition 1.2 and its dual, which is discussed in the paragraph after Example 1.3 in a slightly different form.

Proposition 1.4. *Let $p = p_1 p_2 \cdots p_n$ be a permutation of length n that avoids 132. Let the ordered pairs of words $(u(p), v(p))$ of length n be defined as follows. The ith letter of $u(p)$ is A if p_i is a left-to-right minimum in p, and B otherwise. The ith letter of $v(p)$ is A if the entry of value i is a left-to-right minimum in p, and B otherwise.*

Then the map $r(p) = (u(p), v(p))$ is injective.

Example 1.5. *If $p = 43512$, then $u(p) = AABAB$ and $v(p) = ABAAB$.*

Proposition 1.4 is clearly equivalent to Proposition 1.2 since they both state that a 132-avoiding permutation is completely determined by its set of left-to-right minima, and the positions of those left-to-right minima in the permutation.

We announce the corresponding statement for 213-avoiding permutations, for future reference.

Proposition 1.6. *Let $p = p_1 p_2 \cdots p_n$ be a permutation of length n that avoids 213. Let the ordered pairs of words $(x(p), y(p))$ of length n be defined as follows. The ith letter of $x(p)$ is C if p_i is* not *a right-to-left maximum in p, and D otherwise. The ith letter of $y(p)$ is C if the entry of value i is* not *a right-to-left maximum in p, and D otherwise.*

Then the map $s(p) = (x(p), y(p))$ is injective.

Example 1.7. *If $p = 35412$, then $x(p) = CDDCD$, and $y(p) = CDCDD$.*

2. Coloring entries

The starting point of our proof is the following decomposition of 1324-avoiding permutations, given in [5].

Let $p = p_1 p_2 \cdots p_n$ be a 1324-avoiding permutation, and let us color each entry of p red or blue as we move from left to right, according the following rules.

(1) If coloring p_i red would create a 132-pattern with all red entries, then color p_i blue, and
(2) if there already is a blue entry smaller than p_i, then color p_i blue;
(3) otherwise color p_i red.

It is then proved in [5] that the red entries form a 132-avoiding permutation and the blue entries form a 213-avoiding permutation. From this, it is not difficult to prove that the number of 1324-avoiding n-permutations is less than 16^n. Indeed, there are at most 2^n possibilities for the set of the red entries (the blue entries being the remaining entries), and there are at most 2^n possibilities for the positions in which red entries are placed (the blue entries then must be placed in the remaining positions). Once the set and positions of the k red entries are known, there are $C_k < 4^k$ possibilities for their permutation, just as there are $C_{n-k} < 4^{n-k}$ possibilities for the permutation of the blue entries, completing the proof of the inequality $S_n(1324) < 16^n$.

Remark 2.1. In [5], the coloring introduced above is used in a more general context. However, in this paper we only study 1324-avoiding permutations. It is worth pointing out that in this situation, rule (2) is actually *extraneous*. That is, dropping rule (2) and keeping rules (1) and (3) leads to the very same coloring as rules (1), (2), and (3) do. In order to see this, let p be a 1324-avoiding permutation. Let us start coloring the entries of p from left to right as the rules (1), (2) and (3) specify. Let us assume that there is at least one entry that gets colored blue only because of rule (2). In that case, there is a leftmost entry with that property; let that entry be denoted by x. Then, by the definition of x, there exists an entry y so that $y < x$, the entry y is on the left of x, and y is blue. Furthermore, because x is the leftmost entry that got colored blue only because of rule (2), the entry y got colored blue because of rule (1). That means that there is a 132-pattern acy in which a and c are red. Note that $c < x$ is impossible, since that would mean that $acyx$ is a 1324-pattern. So $y < x < c$, and therefore, acx is a 132-pattern with its first two entries red. That means that x is colored blue by rule (1), a contradiction.

3. Refining the coloring

In this section, we improve the upper bound on $S_n(1324)$ by using a more refined decomposition of 1324-avoiding permutations, which enables us to carry out a more careful counting argument. Let us color each entry of the 1324-avoiding permutation $p = p_1p_2 \cdots p_n$ red or blue as in Section 2. Furthermore, let us mark each entry of p with one of the letters A, B, C, or D as follows.

(1) Mark each red entry that is a left-to-right minimum in the partial permutation of red entries by A,
(2) mark each red entry that is not a left-to-right minimum in the partial permutation of red entries by B,
(3) mark each blue entry that is not a right-to-left maximum in the partial permutation of blue entries by C, and
(4) mark each blue entry that is a right-to-left maximum in the partial permutation of blue entries by D.

Call entries marked by the letter X entries of *type* X. Let $w(p)$ be the n-letter word over the alphabet $\{A, B, C, D\}$ defined above. In other words, the ith letter of $w(p)$ is the type of p_i in p. Let $z(p)$ be the n-letter word over the alphabet $\{A, B, C, D\}$ whose ith letter is the type of the entry of value i in p.

Remark 3.1. *Note that the function* $f(p) = (w(p), z(p))$ *in fact applies the map* r *of Proposition 1.4 to the string* p_{red} *of red entries of* p*, and the map* s *of Proposition 1.6 to the string* p_{blue} *of blue entries of* p*. So given* $f(p) = (w(p), z(p))$*, we can immediately recover* $r(p_{red})$ *and* $s(p_{blue})$*. Indeed,* $r(p_{red})$ *is the pair of subwords of* $w(p)$ *and* $z(p)$ *that consist of letters* A *and* B*, whereas* $s(p_{blue})$ *is the pair of subwords of* $w(p)$ *and* $z(p)$ *that consist of letters* C *and* D*.*

Conversely, if we are given $r(p_{red}) = (u(p_{red}), v(p_{red}))$ *and* $s(p_{blue}) = (x(p_{blue}), y(p_{blue}))$*,* and *we know in which positions of* p *the red entries are, and entries of which value of* p *are red, we can recover* $f(p)$ *as follows. Shuffle the words* $u(p_{red})$ *and* $x(p_{blue})$ *so that letters* A *and* B *are in positions that belong to red entries in* p*, and shuffle the words* $v(p_{red})$ *and* $y(p_{blue})$ *so that letters* A *and* B *are in positions* j *for which the entry of value* j *is red in* p*.*

Example 3.2. *Let* $p = 3612745$*. Then the subsequence of red entries of* p *is* 36127*, the subsequence of blue entries of* p *is* 45*, so* $w(p) = ABABBCD$*, while* $z(p) = ABACDBB$*.*

The following lemma shows a property of $w(p)$ that will enable us to improve the upper bound on $S_n(1324)$. Let us say that a word w has a *CB-factor* if somewhere in w, a letter C is immediately followed by a letter B.

Lemma 3.3. *If* p *is 1324-avoiding, then* $w(p)$ *has no* CB*-factor.*

Proof. Let us assume that in $p = p_1 p_2 \cdots p_n$, the entry p_i is of type C, while the entry p_{i+1} is of type B. That means that $p_i > p_{i+1}$, otherwise the fact that p_i is blue would force p_{i+1} to be blue. Furthermore, since p_i is not a right-to-left maximum, there is an entry d on the right of p_i (and on the right of p_{i+1}) so that $p_i < d$. Similarly, since p_{i+1} is not a left-to-right minimum, there is an entry a on its left so that $a < p_{i+1}$. However, then $ap_i p_{i+1} d$ is a 1324-pattern, which is a contradiction. □

Lemma 3.4. *If* p *is 1324-avoiding, then there is no entry* i *in* p *so that* i *is of type* C *and* $i+1$ *is of type* B*.*

Proof. Analogous to the proof of Lemma 3.3. If such a pair existed, i would have to be on the right of $i+1$, since i is blue and $i+1$ is red. As i is not a right-to-left maximum, there would be a larger entry d on its right. As $i+1$ is not a left-to-right minimum, there would be a smaller entry a on its left. However, then $a(i+1)id$ would be a 1324-pattern. □

Lemma 3.5. *Let* h_n *be the number of words of length* n *that consist of letters* A*,* B*,* C *and* D *that have no* CB*-factors. Then we have*

$$H(x) = \sum_{n \geq 0} h_n x^n = \frac{1}{1 - 4x + x^2}.$$

This implies

$$h_n = \frac{3+2\sqrt{3}}{6} \cdot \left(2+\sqrt{3}\right)^n + \frac{3-2\sqrt{3}}{6} \cdot \left(2-\sqrt{3}\right)^n. \tag{1}$$

Proof. (of Lemma 3.5) Let $\mathcal{H}_n$ denote the set of all words of length n over the alphabet $\{A, B, C, D\}$ that contain no CB-factors. Using the notation from the book *Analytic Combinatorics* by Philippe Flajolet and Robert Sedgewick [6], we claim that if $n \geq 2$, then

$$\mathcal{H}_{n-1} * \{A, B, C, D\} = \mathcal{H}_n + \mathcal{H}_{n-2} * CB. \tag{2}$$

Indeed, let us take a word that is an element of $\mathcal{H}_{n-1}$, and append a letter A, B, C or D to its end. The result is a word that is in $\mathcal{H}_n$, except when the addition of the new letter creates a CB-factor. In that case, that CB-factor at the end of the word is preceded by a word that belongs to $\mathcal{H}_{n-2}$.

Noting that $h_0 = 1$ and $h_1 = 4$, formula (2) leads to the functional equation

$$4xH(x) + 1 = H(x) + x^2H(x).$$

Expressing $H(x)$, we obtain

$$H(x) = \frac{1}{1-4x+x^2}$$

as claimed. In order to find the exact formula for h_n, we use partial fractions. Note that $\alpha = 2+\sqrt{3}$ and $\beta = 2-\sqrt{3}$ are the roots of the denominator of $H(x)$, and also note that $\alpha\beta = 1$. Let us look for real numbers r and s so that

$$H(x) = \frac{r}{1-\alpha x} + \frac{s}{1-\beta x}$$

holds for all real numbers x. Multiplying both sides by $1-4x+x^2$, we get the identity

$$1 = r(1-\beta x) + s(1-\alpha x). \tag{3}$$

As (3) must hold for all real numbers x, it has to hold in particular for $x = \beta = 1/\alpha$. That substitution reduces (3) to $1 = r(1-\beta^2)$, yielding that

$$r = \frac{1}{1-\beta^2} = \frac{3+2\sqrt{3}}{6}.$$

In a similar manner, substituting $x = \alpha = 1/\beta$ in (3) yields $s = \frac{3-2\sqrt{3}}{6}$. Therefore,

$$\begin{aligned} H(x) &= \frac{3+2\sqrt{3}}{6} \cdot \frac{1}{1-\alpha x} + \frac{3-2\sqrt{3}}{6} \cdot \frac{1}{1-\beta x} \\ &= \frac{3+2\sqrt{3}}{6} \cdot \sum_{n\geq 0} \alpha^n x^n + \frac{3-2\sqrt{3}}{6} \cdot \sum_{n\geq 0} \beta^n x^n, \end{aligned}$$

and our claim is proved by equating coefficients of x^n. □

The following, simple but crucial lemma tells us that the ordered pair $(w(p), z(p))$ completely determines the 1324-avoiding permutation p.

Lemma 3.6. *Let $Av_n(1324)$ be the set of all 1324-avoiding n-permutations. Then the map $f : Av_n(1324) \to \mathcal{H}_n \times \mathcal{H}_n$, given by $f(p) = (w(p), z(p))$ is injective.*

Proof. Let $(w, z) \in \mathcal{H}_n \times \mathcal{H}_n$, and let us assume that $f(p) = (w, z)$, that is, that $w(p) = w$, and $z(p) = z$ for some $p \in Av_n(1324)$.

Then w tells us for which indices i the entry p_i will be of type A, namely for the indices i for which the ith letter of w is A. Similary, w tells us the indices j for which the entry p_i is of type B, type C, or type D.

After this, we can use z to figure out which *entries* of p are of type A, type B, type C or type D.

Now let w_{AB} (resp. z_{AB}) be the subword of w (resp. z) that consists of all the letters A and B in w (resp. z). In other words, the pair (w_{AB}, z_{AB}) contains all information about the *red* entries of p. It then follows from Proposition 1.4 that there exists at most one 132-avoiding permutation p' for which $r(p') = (w_{AB}, z_{AB})$.

Define w_{CD} and z_{CD} in an analogous manner. Then Proposition 1.6 shows that there exists at most one 213-avoiding permutation p'' for which $s(p'') = (w_{CD}, z_{CD})$.

It is now immediate from Remark 3.1 that f is injective. Indeed, if $f(p) = (w, z)$, then the red entries of p must form the unique permutation p' for which $r(p') = (w_{AB}, z_{AB})$, and the blue entries of p must form the unique permutation p'' for which $s(p'') = (w_{CD}, z_{CD})$. Finally, as we said in the second and third paragraphs of this proof, the pair (w, z) uniquely determines the set and positions of red entries of p, and the set and positions of blue entries of p. $\square$

We are now ready to announce and prove the main enumeration result of this paper.

Corollary 3.7. *For all positive integers n, the inequality*

$$S_n(1324) < h_{n-1}^2 < \left(2 + \sqrt{3}\right)^{2n} = \left(7 + 4\sqrt{3}\right)^n$$

holds.

Proof. The fact that $S_n(1324) < h_n^2$ is immediate from the injective property of f that we have just proved in Lemma 3.6. In order to complete the proof of the first inequality, note that the image of f consists of ordered pairs $(w(p), z(p))$ in which both $w(p)$ and $z(p)$ starts with an A, since both p_1 and 1 are always red, and left-to-right minima within the string of red entries (and even in all of p). The rest follows from formula (1), since in that formula, the second summand is negative, and in the first summand, the coefficient $(3 + 2\sqrt{3})/6$ is smaller than the base $(2 + \sqrt{3})$. $\square$

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M. Bóna, Department of Mathematics, University of Florida, 358 Little Hall, PO Box 118105, Gainesville, FL 32611–8105 (USA)