THE AVERAGE NUMBER OF BLOCK INTERCHANGES NEEDED TO SORT A PERMUTATION AND A RECENT RESULT OF STANLEY

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Abstract. We use an interesting recent result of probabilistic flavor concerning the product of two permutations consisting of one cycle each to find an explicit formula for the average number of block interchanges needed to sort a permutation of length n.

1. Introduction

1.1. The main definition, and the outline of this paper.

Let $p = p_1p_2\cdots p_n$ be a permutation. A *block interchange* is an operation that interchanges two blocks of consecutive entries without changing the order of entries within each block. The two blocks do not need to be adjacent. Interchanging the blocks $p_ip_{i+1}\cdots p_{i+a}$ and $p_jp_{j+1}\cdots p_{j+b}$ with $i+a<j$ results in the permutation

$$p_1\cdots p_{i-1}p_jp_{j+1}\cdots p_{j+b}p_{i+a+1}\cdots p_{j-1}p_ip_{i+1}\cdots p_{i+a}p_{j+b+1}\cdots p_n.$$

For instance, if $p = 3417562$, then interchanging the block of the first two entries with the block of the last three entries results in the permutation 5621734.

In this paper, we are going to compute the average number of block interchanges to sort a permutation of length n. The methods used in the proof are surprising for several reasons. First, our enumeration problem will lead us to an interesting question on the symmetric group that is very easy to ask and that is of probabilistic flavor. Second, this question then turns out to be surprisingly difficult to answer– the conjectured answer of one of the authors has only recently been proved by Richard Stanley [7], whose proof was not elementary.

1.2. Earlier Results and Further Definitions.

The first significant result on the topic of sorting by block interchanges is by D. A. Christie [3], who provided a direct way of determining the number of block interchanges necessary to sort any given permutation p. The following definition was crucial to his results.

Definition 1. *The* cycle graph $G(p)$ *of the permutation* $p = p_1p_2\cdots p_n$ *is a directed graph on vertex set* $\{0,1,\cdots,n\}$ *and* $2n$ *edges that are colored either black or gray as follows. Set* $p_0 = 0$.

1. *For* $0 \leq i \leq n$, *there is a black edge from* p_i *to* p_{i-1}, *where the indices are to be read modulo* $n+1$, *and*
2. *For* $0 \leq i \leq n$, *there is a gray edge from* i *to* $i+1$, *where the indices are to be read modulo* $n+1$.

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See Figures 1 and 2 for three examples. Black edges are represented by solid, thick lines, and gray edges are represented by thin, dotted lines.

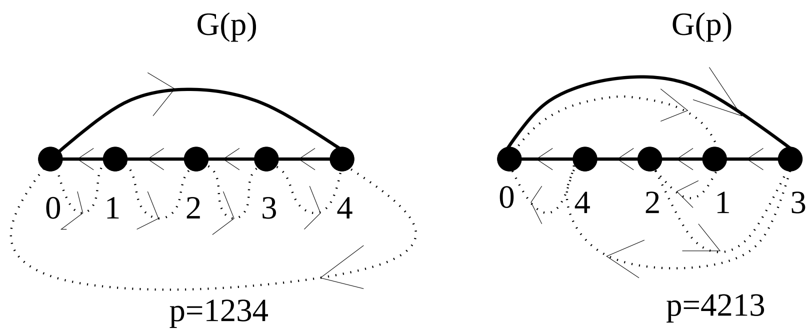

FIGURE 1. The graphs $G(p)$ for $p = 1234$ and $p = 4213$. One sees that $c(G(1234)) = 5$ and $c(G(4213)) = 1$.

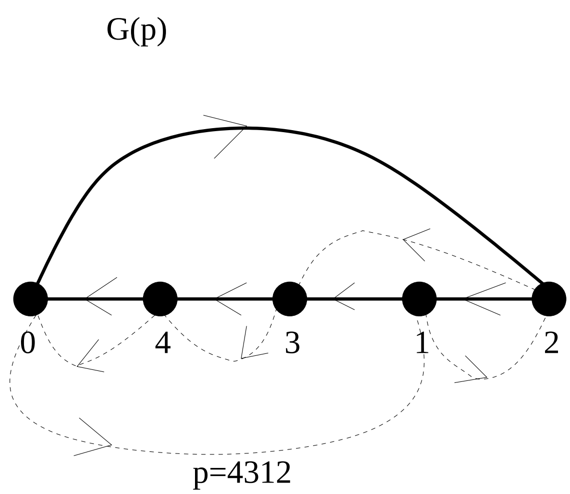

FIGURE 2. The graph $G(p)$ for $p = 4312$. One sees that $c(G(4312)) = 3$.

It is straightforward to show that $G(p)$ has a unique decomposition into edge-disjoint directed cycles in which the colors of the edges alternate. (Just note that each vertex has one edge of each color leaving that vertex, and one edge of each color entering that vertex.) Let $c(G(p))$ be the number of directed cycles in this decomposition of $G(p)$. The main enumerative result of [3] is the following formula. In the rest of this paper, permutations of length n will be called n-permutations, for shortness.

Theorem 1. *The number of block interchanges needed to sort the n-permutation p is $\frac{n+1-c(G(p))}{2}$.*

Note that in particular this implies that $n+1$ and $c(G(p))$ are always of the same parity. Christie has also provided an algorithm that sorts p using $\frac{n+1-c(G(p))}{2}$ block interchanges. As the identity permutation is the only n-permutation that takes zero block interchanges to sort, it is the only n-permutation p satisfying $c(G(p)) = n+1$.

Theorem 1 shows that in order to find the average number A_n of block interchanges needed to sort an n-permutation, we will need the average value of $c(G(p))$ for such permutations. The following definition [4] will be useful.

Definition 2. *The* Hultman number $\mathcal{S}_H(n,k)$ *is the number of n-permutations p satisfying $c(G(p)) = k$.*

The first few Hultman numbers are shown below. The first row assumes $n = 1$, the second row assumes $n = 2$, and so on. The kth element of the nth row is the value $\mathcal{S}_H(n,k)$.

- 0, 1
- 1, 0, 1
- 0, 5, 0, 1
- 8, 0, 15, 0, 1
- 0, 84, 0, 35, 0, 1.

So the Hultman numbers are somewhat analogous to the signless Stirling numbers of the first kind that count n-permutations with k cycles. The name Hultman numbers is justified as Axel Hultman took the initiative in studying these numbers in his master's thesis [5].

This is a good place to point out that in this paper, we will sometimes discuss *cycles of the permutation* p in the traditional sense, which are not to be confused with the *directed cycles of* $G(p)$, counted by $c(G(p))$. Following [4], the number of cycles of the permutation p will be denoted by $c(\Gamma(p))$. Indeed, the cycles of a permutation p are equivalent to the directed cycles of the graph $\Gamma(p)$ in which there is an edge from i to j if $p(i) = j$. For instance, if $p = 1234$, then $c(\Gamma(p)) = 4$, while $c(G(p)) = 5$.

The following recent theorem of Doignon and Labarre [4] brings the Hultman numbers closer to the topic of enumerating permutations according to their cycle structure (in the traditional sense). Let S_n denote the symmetric group of degree n. At this point, we would like to emphasize that we will multiply permutations from left to right. That is, in a product pq, we first apply p, and then q.

Theorem 2. *The Hultman number $\mathcal{S}_H(n,k)$ is equal to the number of ways to obtain the cycle $(12\cdots n(n+1)) \in S_{n+1}$ as a product qr of permutations, where $q \in S_{n+1}$ is any cycle of length $n+1$, and the permutation $r \in S_{n+1}$ has exactly k cycles, that is $c(\Gamma(r)) = k$.*

We note that in [4], permutations are multiplied right-to-left, and so Theorem 2 is proved for products in which we first apply permutation r, then the cyclic permutation q. However, this is clearly equivalent to the above version of the Theorem, since a permutation and its inverse have the same cycle structure.

2. Our Main Result

The following immediate consequence of Theorem 2 is more suitable for our purposes.

Corollary 1. *The Hultman number $\mathcal{S}_H(n,k)$ is equal to the number of $(n+1)$-cycles q so that the product $(12\cdots n(n+1))q$ is a permutation with exactly k cycles, that is, $c(\Gamma((12\cdots n(n+1))q) = k$.*

Proof. If $(12\cdots n(n+1))q = w$, where w has k cycles, and q is an $(n+1)$-cycle, then multiplying both sides of the last equation by q^{-1} from the right, we get the equation

$$(12\cdots n(n+1)) = wq^{-1}.$$

The claim of the Corollary is now immediate from Theorem 2, since q^{-1} is a cycle of length $n+1$. □

Example 1. *For any fixed* n*, we have* $\mathcal{S}_H(n, n+1) = 1$ *since* $c(G(p)) = n+1$ *if and only if* p *is the identity permutation. And indeed, there is exactly one* $(n+1)$*-cycle (in fact, one permutation)* $q \in S_{n+1}$ *so that* $(12..\cdots n(n+1))q$ *has* $n+1$ *cycles, namely* $q = (12\cdots n(n+1))^{-1} = (1(n+1)n\cdots 2)$.

In other words, finding the average of the numbers $c(G(p))$ over all n-permutations p is equivalent to finding the average of the numbers $c(\Gamma((12\cdots n(n+1))q)$, where q is an $(n+1)$-cycle.

Let us consider the product $s = (12\cdots n)z$, where z is a cycle of length n. Let us insert the entry $n+1$ into z to get the permutation z' so that $n+1$ is inserted between two specific entries a and b in the following sense.

$$z'(i) = \begin{cases} z(i) \text{ if } i \notin \{a, n+1\}, \\ n+1 \text{ if } i = a, \text{ and} \\ b \text{ if } i = n+1. \end{cases}$$

See Figure 2 for an illustration.

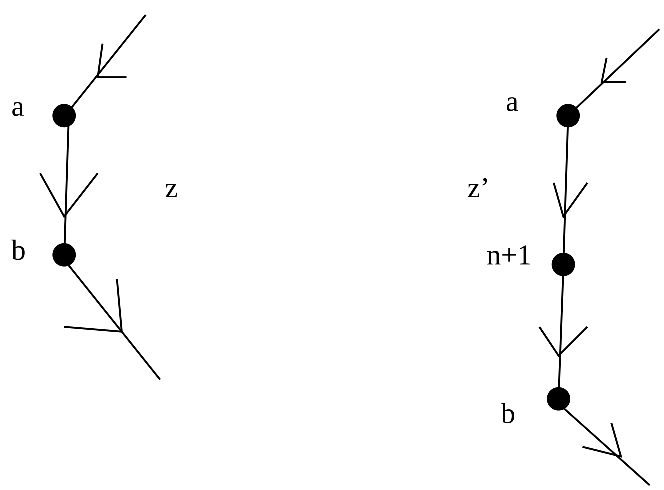

FIGURE 3. How z' is obtained from z.

The following proposition is the first step towards describing how the Hultman numbers grow. It could be deduced from standard facts related to the product of a cycle and a transposition that are present in various textbooks, at least at the level of exercises. However, we will include a proof in order to keep the discussion self-contained.

Proposition 1. *Let a, b, and z' be defined as above, and let $s' = (12\cdots(n+1))z'$. Then we have*

$$c(\Gamma(s')) = \begin{cases} c(\Gamma(s)) - 1 \text{ if } 2 \leq a, \text{ and } a-1 \text{ and } z(1) \text{ are not in the same cycle of } s, \\ c(\Gamma(s)) + 1 \text{ if } 2 \leq a, \text{ and } a-1 \text{ and } z(1) \text{ are in the same cycle of } s, \\ c(\Gamma(s)) + 1 \text{ if } a = 1. \end{cases}$$

Proof. Let us assume first that $a \geq 2$, and that $a-1$ is in a cycle C_1 of s, and $z(1)$ is in a different cycle C_2 of s. Let $C_1 = ((a-1)b\cdots)$ and let $C_2 = (z(1)\cdots n)$. After the insertion of $n+1$ into z, the obtained permutation $s' = (12\cdots(n+1))z'$ sends $a-1$ to $n+1$, then $n+1$ to $z(1)$, then leaves the rest of C_2 unchanged till its last entry. Then it sends n back to $z'(n+1) = b$, from where it continues with the rest of C_1 with no change. So in s', the cycles C_1 and C_2 are united, the entry $n+1$ joins their union, and there is no change to the other cycles of s. See Figure 4 for an illustration.

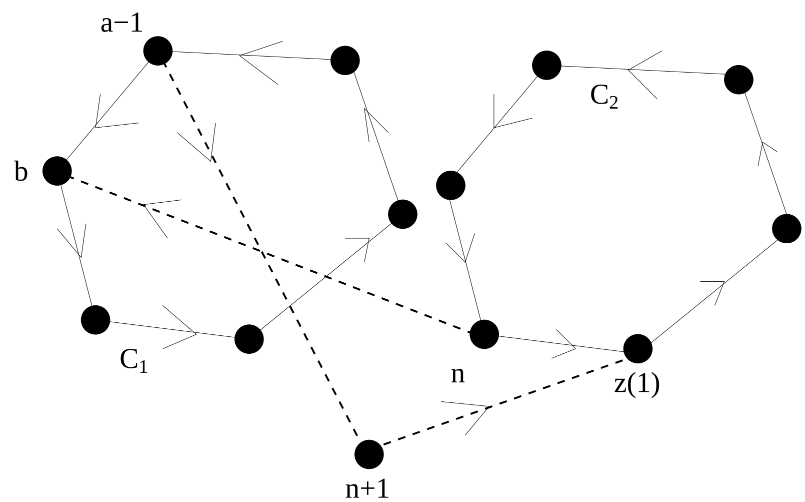

FIGURE 4. If $a-1$ and $z(1)$ are in different cycles of s, those cycles will turn into one.

Let us now assume that $a \geq 2$, and that $a-1$ and $z(1)$ are both in the same cycle C of s. Then $C = ((a-1)b\cdots nz(1)\cdots)$. After the insertion of $n+1$ into z, the obtained permutation $s' = (12..\cdots(n+1)z'$ sends $a-1$ to $n+1$, then $n+1$ to $z(1)$, cutting off the part of C that was between $a-1$ and n. So C is split into two cycles, the cycle $C' = ((a-1)(n+1)z(1)\cdots)$ and the cycle $C'' = (b\cdots n)$. Note that $s'(n) = b$ since $z'(n+1) = b$. See Figure 5 for an illustration.

Finally, if $a = 1$, then $s'(n+1) = (n+1)$, so the entry $n+1$ forms a 1-cycle of s', and the rest of the cycles of s do not change. □

Let TC_n denote the set of ordered pairs (x, y) of n-permutations that consist of one n-cycle each. Then TC_n has $(n-1)!^2$ elements. Let a_n be the average number of cycles of xy, where (x, y) ranges the elements of TC_n.

As Proposition 1 shows, inserting $n+1$ into a position of z will sometimes decrease and sometimes increase the number of cycles of the product s'. The question is, of

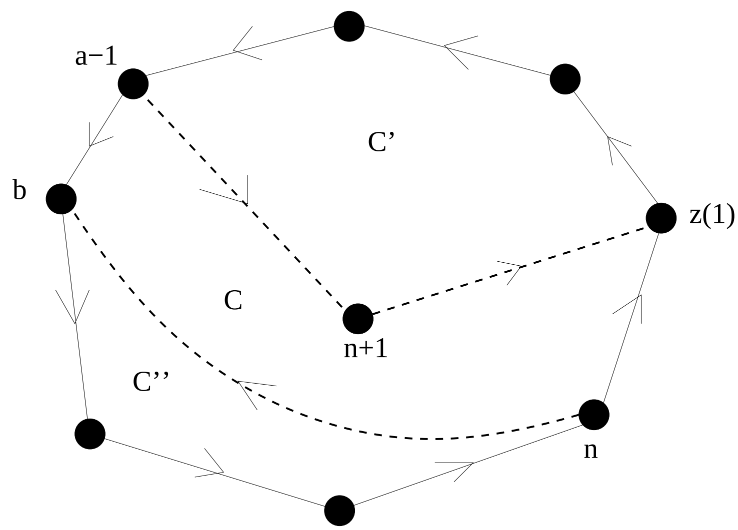

FIGURE 5. If $a-1$ and $z(1)$ are in the same cycle of s, that cycle will split into two cycles.

course, how many times will an increase and how many times will a decrease occur. In light of Proposition 1, this is the same question as asking how often the entries $z(1)$ and $a-1$ are in the same cycle of s. Furthermore, since z is just an arbitrary n-cycle, and $a-1$ is an arbitrary element of a fixed n-cycle, this is equivalent to the following question.

Question 1. *Let i and j be two fixed elements of the set $[n] = \{1, 2, \cdots, n\}$. Select an element (x, y) of TC_n at random. What is the probability that the product xy contains i and j in the same cycle?*

The first author of this article has conjectured that the answer to this question was $1/2$ for odd n. This conjecture was recently proved by Richard Stanley [7], who also settled the question for even values of n.

Theorem 3. [7] *Let i and j be two fixed, distinct elements of the set $[n]$, where $n > 1$. Let (x, y) be a randomly selected element of TC_n. Let $p(n)$ be the probability that i and j are in the same cycle of xy. Then*

$$p(n) = \begin{cases} \frac{1}{2} \text{ if } n \text{ is odd, and} \\ \frac{1}{2} - \frac{2}{(n-1)(n+2)} \text{ if } n \text{ is even.} \end{cases}$$

It is now not difficult to describe how the average of the Hultman numbers grow. Let a_n be the average number of cycles in all permutations of the set $\{xy|(x, y) \in TC_n\}$. (Theorem 2 shows that the a_n is alsothe average value of $\mathcal{S}_H(n-1, k)$, taken for all $(n-1)!$ permutations of length $n-1$.)

Lemma 1. *Let $n \geq 1$. Then the numbers a_n grow as follows.*

(1) *If $n = 2m + 2$, then we have $a_n = a_{n-1} + \frac{1}{n-1}$,*
(2) *If $n = 2m + 1$, then we have $a_n = a_{n-1} + \frac{1}{n-1} - \frac{1}{m(m+1)}$*

Proof. (1) We apply Proposition 1, with n replaced by $n-1$, which is an odd number. So z' is a cycle of length n obtained from a cycle z of length $n-1$

through the insertion of the maximal element n into one of $n-1$ possible positions. If $a \neq 1$, then $a-1$ and $z(1)$ are equally likely to be in the same cycle or not in the same cycle of s. Therefore, an increase of one or a decrease of one in $c(G(s))$ is equally likely. If, on the other hand, $a = 1$, which occurs in $1/(n-1)$ of all cases, then $c(G(s))$ increases by one. So

$$a_n = \frac{n-2}{n-1}a_{n-1} + \frac{1}{n-1}(a_{n-1}+1) = a_{n-1} + \frac{1}{n-1}.$$

(2) We again apply Proposition 1, with n replaced by $n-1$, which is now an even number, namely $n-1 = 2m$. If $a \neq 1$, which happens in $(n-2)/(n-1)$ of all cases, then the probability of $a-1$ and $z(1)$ falling into the same cycle of s is $\frac{1}{2} - \frac{2}{(n-2)(n+1)} = \frac{1}{2} - \frac{1}{(2m-1)(m+1)}$ by Theorem 3. By Proposition 1, in these cases $c(G(s))$ grows by one. If $a = 1$, which occurs in $1/(n-1)$ of all cases, then $c(G(s))$ always grows by one. So

$$\begin{aligned} a_n &= \frac{n-2}{n-1} \cdot \left(\frac{1}{2} - \frac{1}{(2m-1)(m+1)}\right)(a_{n-1}+1) \\ &+ \frac{n-2}{n-1} \cdot \left(\frac{1}{2} + \frac{1}{(2m-1)(m+1)}\right)(a_{n-1}-1) \\ &+ \frac{1}{n-1}(a_{n-1}+1), \end{aligned}$$

which is equivalent to the statement of the lemma as can be seen after routine rearrangements.

□

We are now in position to state and prove our explicit formula for a_n.

Theorem 4. *For all positive integers n, we have*

$$a_n = \frac{1}{\lfloor (n+1)/2 \rfloor} + \sum_{i=1}^{n-1} \frac{1}{i}.$$

Proof. (of Theorem 4) The statement is a direct consequence of Lemma 1 if we note the telescoping sum $\sum_{i=1}^{t} \frac{1}{i(i+1)} = 1 - \frac{1}{t+1}$ obtained when summing the values computed in the second part of Lemma 1. □

Note that it is well-known that on average, an n-permutation has $\sum_{i=1}^{n} \frac{1}{i}$ cycles. This is the average value of $c(\Gamma(p))$ for a randomly selected n-permutation. Theorem 4 shows that a_n differs from this by about $1/n$.

Finally, our main goal is easy to achieve.

Theorem 5. *The average number of block interchanges needed to sort an n-permutation is*

$$b_n = \frac{n - \frac{1}{\lfloor (n+2)/2 \rfloor} - \sum_{i=2}^{n} \frac{1}{i}}{2}.$$

Proof. By Theorem 2 and Theorem 4, the average value of $c(G(p))$ over all permutations p of length n is $a_{n+1} = \frac{1}{\lfloor (n+2)/2 \rfloor} + \sum_{i=1}^{n} \frac{1}{i}$. Our claim now immediately follows from Theorem 1. □

As the sum of the harmonic series $\sum_{i=1}^{n} \frac{1}{i}$ approaches $\log n$ as n goes to infinity, the average number of block interchanges needed to sort an n-permutation is close to $(n - \log n)/2$.

3. Remarks and Further Directions

Richard Stanley's proof of Theorem 3 is not elementary. It uses symmetric functions, exponential generating functions, integrals, and a formula of Boccara [1]. A more combinatorial proof of the stunningly simple answer for the case of odd n would still be interesting.

As pointed out by Richard Stanley [8], there is an alternative way to obtain the result of Theorem 4 without using Theorem 3, but that proof in turn uses symmetric functions and related machinery. It is shown in Exercises 69(a) and 69(c) of [9] that

$$(1) \qquad P_n(q) = \sum_w q^{c(\Gamma(w(12\cdots n))} = \frac{1}{\binom{n+1}{2}} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} c(n+1, n-2i) q^{n-2i},$$

where, as usual, $c(n,k)$ is a signless Stirling number of the first kind, that is, the number of permutations of length n with k cycles, w ranges over all $(n-1))!$ cycles of length n in the symmetric group of degree n, and $w(12\cdots n)$ denotes the product of w and $(12\cdots n)$.

Now a_n can be computed by considering $P_n'(1)$, which in turn can be computed by considering the well-known identity

$$F_{n+1}(x) = \sum_{k=1}^{n+1} c(n+1,k)x^k = x(x+1)\cdots(x+n),$$

and then evaluating $F_{n+1}'(1) + F_{n+1}'(-1)$.

The present paper provides further evidence that the cycles of the graph $G(p)$ have various enumerative properties that are similar to the enumerative properties of the graph $\Gamma(p)$, that is, the cycles of the permutation p. This raises the question as to which well-known properties of the Stirling numbers, such as unimodality, log-concavity, real zeros property, hold for the Hultman numbers as well. (See for instance Chapter 8 of [2] for definitions and basic information on these properties.) A simple modification is necessary since $\mathcal{S}_H(n,k) = 0$ if n and k are of the same parity. So let

$$Q_n(q) = \begin{cases} \sum_{(x,y)\in TC_n} q^{c(\Gamma(xy))/2} \text{ if } n \text{ is even} \\ \sum_{(x,y)\in TC_n} q^{(c(\Gamma(xy))+1)/2} \text{ if } n \text{ is odd.} \end{cases}$$

While the coefficients of $P_n(q)$ are *all* the Hultman numbers $\mathcal{S}_H(n-1,1)$, $\mathcal{S}_H(n-1,2), \cdots, \mathcal{S}_H(n-1,n-1)$, the coefficients of $Q_n(q)$ are the *nonzero* Hultman numbers $\mathcal{S}_H(n-1,k)$.

Clearly, $Q_n(q) = P_n(q^2)$ if n is even, and $Q_n(q) = qP_n(q^2)$ if n is odd. However, Exercise 69(b) of [9] shows that all roots of $P_n(q)$ have real part 0. Hence the roots of $Q_n(q)$ are all real and non-positive, from which the log-concavity and unimodality of the coefficients of $Q_n(q)$ follows. This raises the question of whether there is a combinatorial proof for the latter properties, possibly along the lines of the work of Bruce Sagan [6] for the Stirling numbers of both kinds. Perhaps it is useful to note that (1) and Theorem 2 imply that

$$\mathcal{S}_H(n,k) = \begin{cases} c(n+2,k)/\binom{n+2}{2} \text{ if } n-k \text{ is odd,} \\ 0 \text{ if } n-k \text{ is even.} \end{cases}$$

Finally, to generalize in another direction, we point out that it is very well-known (see, for example, Chapter 4 of [2]), that if we select a n-permutation p at random, and i and j are two fixed, distinct positive integers at most as large as n, then the probability that p contains i and j in the same cycle is $1/2$. Theorem 3 shows that if n is odd, then the multiset $\{xy|(x, y) \in TC_n\}$ behaves just like the set S_n of all permutations in this aspect. This raises the question whether there are other naturally defined subsets (or multisets) of n-permutations in which this phenomenon occurs.

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References

[1] G. Boccara, Nombres de répresentations d'une permutation comme produit de deux cycles de longueur données, *Discrete Math.* **29** (1980), 105-134.
[2] M. Bóna, Introduction to Enumerative Combinatorics, McGraw-Hill, 2007.
[3] D. A. Christie, Sorting Permutations By Block Interchanges, *Inform. Process. Lett.* **60** (1996), 165-169.
[4] J-P. Doignon, A. Labarre, On Hultman Numbers, *Journal of Integer Sequences,* **10** (2007), Article 07.6.2.
[5] A. Hultman, Toric Permutations, *Master's thesis*, Department of Mathematics, KTH, Stockholm, Sweden, 1999.
[6] B. E. Sagan, Inductive and injective proofs of log concavity results. *Discrete Math.* **68** (1988), vol. 2-3, 281–292.
[7] R. Stanley, Two Enumerative Results on Cycles of Permutations, *preprint.*
[8] R. Stanley, personal communication, October, 2008.
[9] R. Stanley, Supplementary Exercises (without solutions) for Chapter 7 (symmetric functions) of Enumerative Combinatorics, Volume 2, *Cambridge University Press*, 1999. Internet resource, available at
`math.mit.edu/~rstan/ec/ch7supp.pdf.`

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