Real Zeros and Partitions without singleton blocks

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Abstract

We prove that the generating polynomials of partitions of an n-element set into non-singleton blocks, counted by the number of blocks, have real roots only and we study the asymptotic behavior of the left-most roots. We apply this information to find the most likely number of blocks. Also, we present a quick way to prove the corresponding statement for cycles of permutations in which each cycle is longer than a given integer r.

1 Introduction

A *partition* of the set $[n] = \{1, 2, \cdots, n\}$ is a set of blocks disjoint blocks $B_1, B_2, \cdots, B_k$ so that $\cup_{i=1}^{k} B_i = [n]$. The number of partitions of $[n]$ into k blocks is denoted by $S(n, k)$ and is called a *Stirling number of the second kind.*

Similarly, the number of permutations of length n with exactly k cycles is denoted by $c(n, k)$, and is called a *signless Stirling number of the first*

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kind. See any textbook on Introductory Combinatorics, such as [2] or [3] for the relevant definitions, or basic facts, on Stirling numbers.

The "horizontal" generating functions, or generating polynomials, of Stirling numbers have many interesting properties. Let n be a fixed positive integer. Then it is well-known (see [2] or [3] for instance) that

$$C_n(x) = \sum_{k=1}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1). \tag{1}$$

In particular, the roots of the generating polynomial $C_n(x)$ are all real (indeed, they are the integers $0, -1, -2, \cdots, -(n-1)$).

Similarly, it is known (see [23], page 20, for instance) that for any fixed positive integer n, the roots of the generating polynomial

$$S_n(x) = \sum_{k=1}^{n} S(n,k)x^k$$

are all real, though they are not nearly as easy to describe as those of $C_n(x)$.

Rodney Canfield [8] (in the case of $r = 1$) and Francesco Brenti [5] (in the general case) have generalized (1) as follows. Let $d_r(n,k)$ be the number of permutations of length n that have k cycles, each longer than r. Such permutations are sometimes called r-derangements. Then the generating polynomial

$$d_{n,r}(x) = \sum_{k \geq 1} d_r(n,k)x^k \tag{2}$$

has real roots only. Present author [4] proved that for any given positive integer constant m, there exists a positive number N so that if $n > N$, then one of these roots will be very close to -1, one will be very close to -2, and so on, with one being very close to $-m$, to close the sequence of m roots being very close to consecutive negative integers.

In this paper, we consider the analogue problem for set partitions. Let $D(n,k)$ be the number of partitions of $[n]$ into k blocks, each consisting of more than one element. We are going to prove that the generating polynomial

$$D_n(x) = \sum_{k \geq 1} D(n,k)x^k \tag{3}$$

has real roots only. We will then use this information to determine the location of the largest coefficient(s) of $D_n(x)$. We also prove that the number of blocks is normally distributed. Finally, we use our methods on r-derangements, and prove the more general result that for any fixed r, the distribution of the number of cycles of r-derangements of length n converges to a normal distribution.

Note that the fact that the two kinds of Stirling numbers behave in the same way under this generalization is not completely expected. Indeed, while $1/e$ of all permutations of length n have no cycles of length 1, and in general, a constant factor of permutations of length n have no cycles of length r or less, the corresponding statement is not true for set partitions. Indeed, almost all partitions of $[n]$ contain a singleton block as we show in Section 3.1. However, as this paper proves, the real zeros property survives.

Finally, we mention that the *vertical* generating functions (minimal block or cycle size is fixed, n varies) of permutations and set partitions have been studied in [8].

2 The Proof of The Real Zeros Property

We start by a recurrence relation satisfied by the numbers $D(n,k)$ of partitions of $[n]$ into k blocks, each block consisting of more than one element. It is straightforward to see that

$$D(n,k) = kD(n-1,k) + (n-1)D(n-2,k-1). \tag{4}$$

Indeed, the first term of the right-hand side counts partitions of $[n]$ into blocks larger than one in which the element n is in a block larger than two, and the second term of the right-hand side counts those in which n is in a block of size exactly two.

Let $D_n(x) = \sum_{k\geq 1} D(n,k)x^k$. Then (4) yields

$$D_n(x) = x\left(D'_{n-1}(x) + (n-1)D_{n-2}(x)\right). \tag{5}$$

Note that $D_1(x) = 0$, and $D_n(x) = x$ if $2 \leq n < 4$. So the first non-trivial polynomial $D_n(x)$ occurs when $n = 4$, and then $D_4(x) = 3x^2 + x$. In the next non-trivial case of $n = 5$, we get $D_5 = 10x^2 + x$.

Theorem 1 *Let $n \geq 2$. Then the polynomial $D_n(x)$ has real roots only. All these roots are simple and non-positive. Furthermore, the roots of $D_n(x)$ and $D_{n-1}(x)$ are interlacing in the following sense. If $D_n(x)$ and $D_{n-1}(x)$ are both of degree d, and their roots are, respectively, $0 = x_0 > x_1 > \cdots > x_{d-1}$, and $0 = y_0 > y_1 > \cdots > y_{d-1}$, then*

$$0 > x_1 > y_1 > x_2 > y_2 > \cdots > x_{d-1} > y_{d-1}, \tag{6}$$

while if $D_n(x)$ *is of degree* $d+1$ *and* $D_{n-1}(x)$ *is of degree* d*, and their roots are, respectively,* $0 = x_0 > x_1 > \cdots > x_d$*, and* $0 = y_0 > y_1 > \cdots > y_{d-1}$*, then*

$$0 > x_1 > y_1 > x_2 > y_2 > \cdots > x_{d-1} > y_{d-1} > x_d. \quad (7)$$

Proof: We prove our statements by induction on n. For $n \leq 4$, the statements are true. Now assume that the statement is true for $n-1$, and let us prove it for n. Let $0 = y_0 > y_1 > \cdots > y_{d-1}$ be the roots of $D_{n-1}(x)$.

First we claim that if $0 > x > y_1$, then $D_{n-1}(x) < 0$, that is, the polynomial D_{n-1} is negative between its two largest roots. Indeed, $D'_{n-1}(0) = D(n-1,1) = 1$, so $D_{n-1}(x)' > 0$ in a neighborhood of 0. This implies that in that neighborhood, $D_{n-1}(x)$ is monotone increasing. As $D_{n-1}(0) = 0$, this implies our claim.

Now consider (5) at $x = y_1$. We claim that at that root, we have both $D'_{n-1}(y_1) < 0$ and $D_{n-2}(y_1) < 0$. The latter is a direct consequence of the previous paragraph and the induction hypothesis. Indeed, the induction hypothesis shows that if z_1 is the largest negative root of D_{n-2}, then $z_1 < y_1 < 0$, and the previous paragraph, applied to D_{n-2} instead of D_{n-1} shows that $D_{n-2}(x) < 0$ if $x \in (z_1, 0)$. So in particular $D_{n-2}(y_1) < 0$. The former follows from Rolle's theorem. Indeed, by Rolle's theorem, between two consecutive roots of D_{n-1}, there has to be a root of D'_{n-1}. As D_{n-1} has simple roots only, say d of them, D'_{n-1} must have $d-1$ simple roots, and therefore, by the pigeon-hole principle, there must be exactly one of them between any two consecutive roots of D_{n-1}. In particular, there is exactly one root of D'_{n-1} between 0 and y_1, so the sign of $D'_{n-1}(y_1)$ is the opposite of the sign of $D'_{n-1}(0) = 1$, that is, it is negative.

So when $x = y_1$, the argument of the previous paragraph shows that the right-hand side of (5) is the product of the negative real number y_1, and the negative real number $D'_{n-1}(y_1) + (n-1)D_{n-2}(y_1)$. Therefore, the left-hand side must be positive, that is, $D_n(y_1) > 0$. As $D_n(x) < 0$ in a neighborhood of 0, this shows that D_n has a root in the interval $(y_1, 0)$.

More generally, we claim that $D_n(x)$ has a root in the interval (y_{i+1}, y_i). For this, it suffices to show that $D_n(y_i)$ and $D_n(y_{i+1})$ have opposite signs. This will follow by (5) if we can show all of the following.

(i) $D'_{n-1}(y_i)$ and $D'_{n-1}(y_{i+1})$ have opposite signs,

(ii) $D_{n-2}(y_i)$ and $D_{n-2}(y_{i+1})$ have opposite signs, and

(iii) $D'_{n-1}(y_i)$ and $D_{n-2}(y_i)$ have equal signs.

Just as before, (i) follows from Rolle's theorem, and (ii) follows from the induction hypothesis. In order to see (iii), note that by Rolle's theorem, D'_{n-1} changes signs exactly i times in $(y_i, 0)$, while by the induction hypothesis, D_{n-2} changes signs $i-1$ times in $(y_i, 0)$. We have seen at the beginning of this proof that in a small neighborhood of 0, D'_{n-1} is positive, while D_{n-2} is negative, so (iii) follows. Therefore, by (5), $D_n(y_i)$ and $D_n(y_{i+1})$ have opposite signs, and so $D_n(x)$ has a root in (y_{i+1}, y_i).

Note that this argument does not *directly* prove that D_n has *exactly* one root in (y_{i+1}, y_i), but it does prove that it has an odd number of roots in each such interval. Indeed, D_n has opposite signs at the endpoints of (y_{i+1}, y_i), it has an odd number of sign changes, and so, an odd number of roots on that interval. As the total number of roots of D_n is at most one larger than that of D_{n-1}, it follows that D_n has indeed exactly one root in each interval (y_{i+1}, y_i).

The above argument completes the proof of the theorem for odd n. When n is even, then D_n is of degree $d+1$, while D_{n-1} is of degree d. In that case, we still have to show that D_n has a root in the interval $(-\infty, y_{d-1})$. However, this follows from the previous paragraph since the last root x_d of D_n must be negative, and cannot be in any of the intervals (y_{i+1}, y_i). $\diamond$

It follows from Theorem 1 that both the sequence $D_4, D_6, D_8, \cdots$, and the sequence $D_5, D_7, D_9, \cdots$ are *Sturm sequences*. The interested reader should consult [22] for the definition and properties of Sturm sequences.

3 Applications of The Real Zeros Property

In this Section, we consider two applications of the real zeros property. Both are combinatorial with a probabilistic flavor.

3.1 Locating peaks

If a polynomial $\sum_{k=1}^{n} b_k x^k$ with positive coefficients has real roots only, then it is known [3] that the sequence $b_1, b_2, \cdots b_n$ of its coefficients is *strongly log-concave*. That is, for all indices $2 \leq j \leq n-1$, the inequality

$$b_j^2 \geq b_{j-1} b_{j+1} \frac{j+1}{j} \cdot \frac{n-j+1}{n-j}$$

holds. In other words, the ratio b_{j+1}/b_j is strictly decreasing with j, and therefore there is at most one index j so that $b_{j+1}/b_j = 1$. Thus the sequence $b_1, b_2, \cdots b_n$ has either one peak, or two consecutive peaks.

A useful tool in finding the location of this peak is the following theorem of Darroch.

Theorem 2 *[11] Let $A(x) = \sum_{k=1}^{n} a_k x^k$ be a polynomial that has real roots only that satisfies $A(1) > 0$. Let m be a peak for the sequence of the coefficients of $A(x)$. Let $\mu = A'(1)/A(1) = \frac{\sum_{k=1}^{n} k a_k}{\sum_{k=1}^{n} a_k}$. Then we have*

$$|\mu - m| < 1.$$

Note that in a combinatorial setup, μ is the average value of the statistic counted by the generating polynomial $A(x)$. For instance, if $A(x) = S_n(x)$, then μ is the average number of blocks in a randomly selected partition of $[n]$.

There is a very extensive list of results on the peak (or two peaks) of the sequence $S(n,1), S(n,2), \cdots, S(n,n)$ of Stirling numbers of the second kind. See [9] for a brief history of this topic and the relevant references. In particular, if $K(n)$ denotes the index of this peak (or the one that comes first, if there are two of them), then $K(n) \sim n/\log n$. More precisely, let r be the unique positive root of the equation

$$re^r = n. \tag{8}$$

Then, for n sufficiently large, $K(n)$ is one of the two integers that are closest to $e^r - 1$. In view of Theorem 2, one way to approach this problem is by computing the average number of blocks in a randomly selected partition of $[n]$.

Now that we have proved that the generating polynomial $D_n(x) = \sum_{k \geq 1} D(n,k)x^k$ has real roots only, it is natural to ask how much of the long list of results on Stirling numbers can be generalized to the numbers $D(n,k)$. In this paper, we will show a quick way of estimating the average number of blocks in a partition of $[n]$ with no singleton blocks, and so, by Darroch's theorem, the location of the peak(s) in the sequence $D(n,1), D(n,2), \cdots, D(n, \lfloor n/2 \rfloor)$. For shortness, let us introduce the notation $D(n) = \sum_k D(n,k)$.

Proposition 1 *Let X_n be the random variable counting blocks of partitions of $[n]$ that have no singleton blocks. Then for all positive integers $n \geq 2$, the equality*

$$E(X_n) = \frac{D(n+1)}{D(n)} - \frac{n(D(n-1))}{D(n)} \tag{9}$$

holds, where $E(X_n)$ denotes the expectation of X_n.

Proof: The total number of blocks in all partitions counted by $D(n)$ is clearly $\sum_{k\geq 1} kD(n,k)$. On the other hand,

$$D(n+1) = \sum_{k\geq 1} kD(n,k) + nD(n-1),$$

since the first term of the right-hand side counts partitions in which the element $n+1$ is in a block of size three or more, and the second term counts partitions in which the element $n+1$ is in a block of size two. So $\sum_{k\geq 1} kD(n,k) = D(n+1) - nD(n-1)$, and the statement follows. $\diamond$

So the peak of the sequence $D(n,1), D(n,2), \cdots$ is one of the two integers bracketing

$$\frac{D(n+1)}{D(n)} - \frac{n(D(n-1))}{D(n)}. \tag{10}$$

We can compare this number with the location $K(n)$ of the peak of the sequence $S(n,1), S(n,2), \cdots, S(n,n)$ as follows.

Let $B(n)$ denote the number of all partitions of $[n]$. This number is often called a *Bell number*. There are numerous precise results on the asymptotics of the Bell numbers. We will only need the following fact [12].

$$\log B(n) = n\left(\log n - \log\log n + O(1)\right), \tag{11}$$

and its consequence that

$$\frac{B(n)}{B(n-1)} \sim \frac{n}{e\log n}. \tag{12}$$

Let Y_n be the random variable that counts blocks of unrestricted partitions of $[n]$, and let S_n be the variable that counts *singleton* blocks of unrestricted partitions of $[n]$. As the average number of blocks in unrestricted partitions of $[n]$ is $\frac{1}{B_n}\sum_{k=1}^{n} kS(n,k) = \frac{B(n+1)-B(n)}{B_n}$, we have

$$E(Y_n) = \frac{B(n+1)}{B(n)} - 1 \sim \frac{n}{e\log n}. \tag{13}$$

Before comparing formulae (9) and (13), we mentioned some simple facts.

For any given element $i \in [n]$, the probability that in a randomly selected unrestricted partition of $[n]$, the element i forms a singleton block is $\frac{B(n-1)}{B(n)}$. Therefore, by linearity of expectation, we have

$$E(S_n) = n\frac{B(n-1)}{B(n)} \sim e\log n. \tag{14}$$

The following simple and well-known result will be very useful for us, and therefore, we state it as a proposition.

Proposition 2 *For all positive integers n, the equality*

$$B(n) = D(n) + D(n+1)$$

holds.

Proof: We define a simple bijection f from the set of partitions of $[n]$ and $[n+1]$ with no singleton blocks into the set of partitions of $[n]$. On partitions counted by $D(n)$, let f act as the identity map. On partitions counted by $D(n+1)$, let f act by removing the element $n+1$ and turning each element that shared a block with $n+1$ into a singleton block. $\diamond$

This simple fact has two important corollaries that we will use.

Corollary 1 *We have $D(n+1) \sim B(n)$.*

Proof: Note that $\frac{B(n)}{D(n)} \to \infty$ since $D(n+1) < B(n)$ and $\frac{D(n+1)}{D(n)} \to \infty$. To see the latter, note that for any h, there exists an N so that if $n > N$, then almost all partitions counted by $D(n)$ have more than h blocks. $\diamond$

Now we can easily see that the locations of the peaks of the sequences $D(n+1,1), D(n+1,2), \cdots$, and $B(n,1), B(n,2), \cdots$, as well as the averages $E(X_{n+1})$ and $E(Y_n)$ as given in formulae (9) and (13) are indeed very close to each other. In fact, $E(X_{n+1}) \sim E(Y_n)$ as can be seen by comparing (9) and (13). We will not attempt a more precise comparison here. However, we would like to point out that the $-\frac{(n+1)(D(n)}{D(n+1)}$ summand in (9), when n is replaced by $n+1$, asymptotically agrees with $E(S_n)$ as computed in (14). This is in line with what one would intuitively expect, since the difference between partitions on which X_{n+1} is defined and partitions on which Y_n is defined is that in the former, singleton blocks are not allowed.

Corollary 2 *Let N_n be the variable counting the non-singleton blocks of a randomly selected unrestricted partition of $[n]$. Let X_n denote the number of blocks of a randomly selected partition of $[n]$ with no singleton blocks.*

Then we have

$$E(X_{n+1} - 1)\frac{D(n+1)}{B(n)} + E(X_n)\frac{D(n)}{B(n)} = E(N_n), \tag{15}$$

and also,

$$E((X_{n+1}-1)^2)\frac{D(n+1)}{B(n)}+E(X_n^2)\frac{D(n)}{B(n)}=E(N_n^2). \qquad (16)$$

Proof: Direct consequence of the bijection f defined in the proof of Proposition 2. $\diamond$

3.2 Another way to locate the peaks

From the results of the previous section – see (10), (11), (12) and Corollary 1 – it follows that the asymptotic location of the K_n^* peak of the sequence $D(n,1), D(n,2), \dots, D(n,n)$ is

$$K_n^* \sim \frac{n}{\log(n)}, \qquad (17)$$

which is the same as for the classical Stirling numbers [13, 15, 17].

Analyzing the ordinary generating function of $D(n,k)$ with the saddle point method we are going to show that as n goes to infinity,

$$D(n,k) \sim S(n,k) \sim \frac{k^n}{k!} \qquad (18)$$

for any fixed k. From this it will follow at once that the asymptotics (17) holds. What is more, relying on (18) and following the proof presented in [16] and in its references one can prove that the maximizing index is close to

$$\frac{n-\frac{1}{2}}{W\left(n-\frac{1}{2}\right)},$$

where $W(n)$ is the Lambert function and it is the unique solution of the equation $W(n)e^{W(n)} = n$.

To prove (18) we need the following proposition.

Proposition 3 *For any fixed integers k, let*

$$f_k(x) = \sum_{n=0}^{\infty} D(n,k)x^n$$

be the ordinary generating function of the sequence $(D(n,k))$. The following recursion holds true:

$$f_k(x) = \frac{x^2}{1-kx}\left(xf_{k-1}(x)\right)', \quad (k \geq 2)$$

with

$$f_1(x) = \frac{x^2}{1-x}.$$

Proof: From (4) it follows that

$$f_k(x) = \sum_{n=0}^{\infty} D(n,k)x^n =$$

$$kx \sum_{n=1}^{\infty} D(n-1,k)x^{n-1} + x^2 \sum_{n=2}^{\infty} (n-1)D(n-2,k-1)x^{n-2} =$$

$$kx \sum_{n=0}^{\infty} D(n,k)x^n + x^2 \sum_{n=2}^{\infty} (n+1)D(n,k-1)x^n =$$

$$kxf_k(x) + x^2 \left(xf_{k-1}'(x) + f_{k-1}(x) \right)$$

which is equivalent to our recursion. The form of $f_1(x)$ is obvious, since $D(n,k) = 1$ if $n \geq 2$. $\diamond$

This simple observation with induction shows that, in general, the $f_k(x)$ functions are rational functions of the form

$$f_k(x) = x^{2k} \frac{p_k(x)}{(kx-1)((k-1)x-1)^2((k-2)x-1)^3 \cdots (x-1)^k} \quad (k \geq 1), \tag{19}$$

where $p_k(x)$ is a polynomial of degree $\frac{k(k-1)}{2}$. These polynomials firstly appeared in a 1934 paper of Ward [21] who studied the representations of the classical Stirling numbers as sums of factorials (see [10] for more details and other citations). The first $f_k(x)$ functions are as follows:

$$f_1(x) = x^2 \frac{-1}{x-1}$$

$$f_2(x) = x^4 \frac{2x-3}{(x-1)^2(2x-1)}$$

$$f_3(x) = x^6 \frac{-12x^3 + 40x^2 - 45x + 15}{(x-1)^3(2x-1)^2(3x-1)}$$

$$f_4(x) = x^8 \frac{288x^6 - 1560x^5 + 3500x^4 - 4130x^3 + 2625x^2 - 840x + 105}{(x-1)^4(2x-1)^3(3x-1)^2(4x-1)}.$$

Going back to our original goal, formula (19) enables us to prove the following.

Proposition 4 *For any fixed positive integer k and large n we have that*

$$D(n,k) = \frac{k^n}{k!} + O(k-1+\varepsilon)^n$$

holds for arbitrary $\varepsilon > 0$.

Proof: The asymptotics of $D(n,k)$ can be determined by analyzing the singularities of its generating function. This is the well known saddle point method described in details by Wilf in [23].

The function $f_k(x)$ has k singular points on the real line and the smallest one is at $x_0 = \frac{1}{k}$. This pole is of order one. The principal part of $f_k(x)$ around this point is

$$PP\left(f_k, \frac{1}{k}\right) = -\frac{1}{k \cdot k!\left(x - \frac{1}{k}\right)}.$$

Since x_0 is a first order pole and there are no more poles with the same absolute value, the saddle point method [23, Theorem 5.2.1] in this particular case says that

$$D(n,k) = [x^n]PP\left(f_k, \frac{1}{k}\right) + O\left(\frac{1}{R'} + \varepsilon\right)^n,$$

where R' is the modulus of the second smallest singular point. In this case this point is $x_1 = \frac{1}{k-1}$. Expanding the $PP\left(f_k, \frac{1}{k}\right)$ principal part with respect to x we get the statement. $\diamond$

3.3 The asymptotics of the zeros of $D_n(x)$

Having proven that the zeros of the $D_n(x)$ polynomials are all real (and negative), it can be asked that *how large* is the leftmost zero of $D_n(x)$? Let z_n^* denote this *leftmost* zero. We point out that an easily calculable upper bound can be given, and this upper bound approximates z_n^* surprisingly well. This approximation is based on a theorem of Laguerre and Samuelson.

Let

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \tag{20}$$

be an arbitrary polynomial such that $a_n \neq 0$. Samuelson [20] – rediscovering the results of Laguerre [14] – gave bounds for the interval which contains all the zeros of a polynomial (20) whose zeros are all *real*. Samuelson's result states that all these zeros are contained in the interval $[x_-, x_+]$, where

$$x_\pm = -\frac{a_1}{n} \pm \frac{n-1}{n}\sqrt{a_1^2 - \frac{2n}{n-1}a_2} \tag{21}$$

for (20).

We want to determine x_- when $p(x) = D_n(x)$ (obviously, $x_+ = 0$). It can easily be seen that

$$a_1 = \frac{D\left(n, \lfloor \frac{n}{2} \rfloor - 1\right)}{D\left(n, \lfloor \frac{n}{2} \rfloor\right)}, \quad \text{and} \quad a_2 = \frac{D\left(n, \lfloor \frac{n}{2} \rfloor - 2\right)}{D\left(n, \lfloor \frac{n}{2} \rfloor\right)}.$$

Hence

$$|z_n^*| \leq -\frac{D\left(n, \lfloor \frac{n}{2} \rfloor - 1\right)}{nD\left(n, \lfloor \frac{n}{2} \rfloor\right)} - \frac{n-1}{n}\sqrt{\left(\frac{D\left(n, \lfloor \frac{n}{2} \rfloor - 1\right)}{D\left(n, \lfloor \frac{n}{2} \rfloor\right)}\right)^2 - \frac{2n}{n-1}\frac{D\left(n, \lfloor \frac{n}{2} \rfloor - 2\right)}{D\left(n, \lfloor \frac{n}{2} \rfloor\right)}}.$$

In the particular case of the $D_n(x)$ polynomials the Samuelson estimation works very well. The following table shows compares the actual values of $|z_n^*|$ with the estimates obtained by the Laguerre-Samuelson theorem for some even numbers n.

n	10	100	200
Numerical value of $\lvert z_n^*\rvert$	9.22	11 085.5	89 360.6
Estimate of Samuelson	9.24	11 163.3	90 022.0

The following table contains the analogous information for odd numbers n.

n	11	101	201
Numerical value of $\lvert z_n^*\rvert$	2.828	2 852.96	22 677.2
Estimate of Samuelson	2.855	2 962.21	23 570.6

By simple combinatorial arguments one can find the special values of $D\left(n, \lfloor \frac{n}{2} \rfloor\right)$, $D\left(n, \lfloor \frac{n}{2} \rfloor - 1\right)$, and $D\left(n, \lfloor \frac{n}{2} \rfloor - 2\right)$ easily. For example,

$$D\left(n, \left\lfloor \frac{n}{2} \right\rfloor\right) = \begin{cases} \frac{n!}{2^{n/2}\left(\frac{n}{2}\right)!}, & \text{if } n \text{ is even;} \\ \binom{n}{3}\frac{(n-3)!}{2^{(n-3)/2}\left(\frac{n-3}{2}\right)!}, & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

With these special values one can find that for the Samuelson estimate

$$x_- \sim -\frac{1}{36\sqrt{6}}n^3 \quad (n \to \infty \text{ is even}),$$

$$x_- \sim -\frac{1}{108\sqrt{10}}n^3 \quad (n \to \infty \text{ is odd}).$$

These asymptotics and the numerical calculations suggest the following conjecture about the asymptotic behavior of the leftmost zeros of $D_n(x)$:

$$z_n^* \sim -c_{\text{even}}n^3 \quad (n \to \infty \text{ is even}),$$

$$z_n^* \sim -c_{\text{odd}}n^3 \quad (n \to \infty \text{ is odd}).$$

4 Basic modularity properties of $D(n,k)$ and $D_n(1)$

A simple application of the binomial theorem (see (24) below) reveals that the well known modularity property

$$S(p,k) \equiv 0 \pmod{p} \quad (1 < k < n) \tag{22}$$

of the Stirling numbers can be transferred to the $D(n,k)$ numbers:

$$D(p,k) \equiv 0 \pmod{p} \quad (1 < k < n). \tag{23}$$

Here p is an arbitrary prime. The most basic Bell number divisibility follows directly from (22):

$$B_p \equiv 2 \pmod{p}$$

for odd primes p. The corresponding divisibility for $D_n = D_n(1)$ is the consequence of (23) and of (3):

$$D_p \equiv 1 \pmod{p}$$

for any prime p including $p = 2$.

The identity in question reads as

$$D(n,k) = \sum_{s=n-k}^{n} \binom{n}{s} (-1)^{n-s} S(s, s+k-n). \tag{24}$$

From this and from the fact that $p \,\big|\, \binom{p}{k}$ $(1 \le k < p)$ (23) follows, indeed.

Formula (24) can be proven easily considering the exponential generating function

$$\sum_{n=0}^{\infty} D(n,k) \frac{x^n}{n!} = \frac{1}{k!} \left(e^x - 1 - x\right)^k.$$

Expanding $((e^x - 1) - x)^k$ with the binomial theorem and using the fact that $\frac{1}{k!}(e^x - 1)^k$ is the exponential generating function of the Stirling numbers, we are done.

We remark that (24) can be generalized to

$$D_m(n,k) = \sum_{s=0}^{n} \binom{n}{s} \sum_{i=0}^{k} (-1)^{k-i} S(s,i) D_{m-1}(n-s, k-i),$$

where $D_m(n,k)$ counts the partitions of $[n]$ into k blocks such that all of the blocks contain at least m elements (so $D(n,k) = D_2(n,k)$).

5 Further Directions

It is natural to ask whether Theorem 1 can be generalized to partitions with all blocks larger than r, where r is a given positive integer. That would parallel the result (2) of Brenti [5] on permutations.

A consequence of the fact that the polynomials $D_n(x)$ have real zeros is that for any fixed n, the sequence $D(n,1), D(n,2), \cdots, D(n, \lfloor n/r \rfloor)$ is log-concave. In [19], Bruce Sagan provides a proof for the special case of $r = 0$, that is, that of the classic Stirling numbers of the second kind. However, his injection proving that result does not preserve the no-singleton-block property. Now that we know that the statement is true, it is natural to ask for an injective proof. Similarly, as we know that $d_{n,r}(x)$ has real roots only (see (1)), we can ask for a combinatorial proof for the fact that the sequence $d_r(n,1), d_r(n,2), \cdots, d_r(n, \lfloor n/r \rfloor)$ is log-concave.

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