On Three Different Notions of Monotone Subsequences

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Abstract

We review how the monotone pattern compares to other patterns in terms of enumerative results on pattern avoiding permutations. We consider three natural definitions of pattern avoidance, give an overview of classic and recent formulas, and provide some new results related to limiting distributions.

1 Introduction

Monotone subsequences in a permutation $p = p_1 p_2 \cdots p_n$ have been the subject of vigorous research for over sixty years. In this paper, we will review three different lines of work. In all of them, we will consider increasing subsequences of a permutation of length n that have a *fixed* length k. This is in contrast to another line of work, started by Ulam more than sixty years ago, in which the distribution of the *longest* increasing subsequence of a random permutation has been studied. That direction of research has recently reached a high point in the article [4] of Baik, Deift and Johansson.

The three directions we consider are distinguished by their definition of monotone subsequences. We can simply require that k entries of a permutation increase from left to right, or we can in addition require that these k entries be in consecutive positions, or we can even require that they be consecutive integers *and* be in consecutive positions.

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2 Monotone Subsequences with No Restrictions

The classic definition of pattern avoidance for permutations is as follows. Let $p = p_1p_2\cdots p_n$ be a permutation, let $k < n$, and let $q = q_1q_2\cdots q_k$ be another permutation. We say that p *contains* q as a pattern if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ so that for all indices j and r, the inequality $q_j < q_r$ holds if and only if the inequality $p_{i_j} < p_{i_r}$ holds. If p does not contain q, then we say that p *avoids* q. In other words, p contains q if p has a subsequence of entries, not necessarily in consecutive positions, which relate to each other the same way as the entries of q do.

Example 1 *The permutation 3174625 contains the pattern 123. Indeed, consider the first, fourth, and seventh entries.*

As in this paper, the monotone pattern $12\cdots k$ plays a special role, we introduce the special notation

$$\alpha_k = 12\cdots k. \tag{1}$$

In particular, p contains α_k if and only if p contains an increasing subsequence of length k. The elements of this increasing subsequence do not have to be in consecutive positions.

The enumeration of permutations avoiding a given pattern is a fascinating subject. Let $S_n(q)$ denote the number of permutations of length n (or, in what follows, n-permutations) that avoid the pattern q.

2.1 Patterns of Length Three

Among patterns of length three, there is no difference between the monotone pattern and other patterns as far as $S_n(q)$ is concerned. This is the content of our first theorem.

Theorem 1 *Let q be any pattern of length three, and let n be any positive integer. Then $S_n(q) = C_n = \binom{2n}{n}/(n+1)$. In other words, $S_n(q)$ is the nth Catalan number.*

Proof: If p avoids q, then the reverse of p avoids the reverse of q, and the complement of p avoids the complement of q. Therefore, $S_n(123) = S_n(321)$ and $S_n(132) = S_n(231) = S_n(213) = S_n(312)$.

The fact that $S_n(132) = S_n(123)$ is proved using the well-known Simion-Schmidt bijection [26]. In a permutation, let us call an entry a *left-to-right* minimum if it is smaller than every entry on its left. For instance, the left-to-right minima of 4537612 are the entries 4, 3, and 1.

Take an n-permutation p of length n that avoids 132, keep its left-to-right minima fixed, and arrange all other entries in decreasing order in the positions that do not belong to left-to-right minima, to get the permutation $f(p)$. For instance, if $p = 34125$, then $f(p) = 35142$. Then $f(p)$ is a union of two decreasing sequences, so it is 123-avoiding. Furthermore, f is a bijection between the two relevant set of permutations. Indeed, if r is a permutation counted by $S_n(123)$, then $f^{-1}(r)$ is obtained by keeping the left-to-right minima of r fixed, and rearranging the remaining entries so that moving from left to right, each slot is filled by the smallest remaining entry that is larger than the closest left-to-right minimum on the left of that position.

In order to prove that $S_n(132) = C_n$, just note that in a 132-avoiding n-permutation, any entry to the left of n must be smaller than any entry to the right of n. Therefore, if n is in the ith position, then there are $S_{i-1}(132)S_{n-i}(132)$ permutations of length n that avoid 132. Summing over all i, we get the recurrence

$$S_n(132) = \sum_{i=0}^{n-1} S_{i-1}(132)S_{n-i}(132),$$

which is the well-known recurrence for Catalan numbers. $\diamond$

2.2 Patterns of Length Four

When we move to longer patterns, the situation becomes much more complicated and less well understood. In his doctoral thesis [30], Julian West published the following numerical evidence.

- for $S_n(1342)$, and $n = 1, 2, \cdots, 8$, we have 1, 2, 6, 23, 103, 512, 2740, 15485
- for $S_n(1234)$, and $n = 1, 2, \cdots, 8$, we have 1, 2, 6, 23, 103, 513, 2761, 15767
- for $S_n(1324)$, and $n = 1, 2, \cdots, 8$, we have 1, 2, 6, 23, 103, 513, 2762, 15793.

These data are startling for at least two reasons. First, the numbers $S_n(q)$ are no longer independent of q; there are some patterns of length four that are easier to avoid than others. Second, the monotone pattern 1234, special as it is, does not provide the minimum or the maximum value for $S_n(q)$. We point out that for each q of the other 21 patterns of length four, it is known that the sequence $S_n(q)$ is identical to one of the three sequences $S_n(1342)$, $S_n(1234)$, and $S_n(1324)$. See [7], Chapter 4, for more details.

Exact formulas are known for two of the above three sequences. For the monotone pattern, Ira Gessel gave a formula using symmetric functions.

Theorem 2 *[16], [15] For all positive integers n, the identity*

$$S_n(1234) = 2 \cdot \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k}^2 \frac{3k^2 + 2k + 1 - n - 2nk}{(k+1)^2 (k+2)(n-k+1)} \tag{2}$$

$$= \frac{1}{(n+1)^2 (n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}. \tag{3}$$

The formula for $S_n(1342)$ is due to the present author [5], and is quite surprising.

Theorem 3 *For all positive integers n, we have*

$$\begin{aligned} S_n(1342) &= (-1)^{n-1} \cdot \frac{(7n^2 - 3n - 2)}{2} \\ &+ 3 \sum_{i=2}^{n} (-1)^{n-i} \cdot 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \cdot \binom{n-i+2}{2}. \end{aligned}$$

This result is unexpected for two reasons. First, it shows that $S_n(1342)$ is not simply less than $S_n(1234)$ for every $n \geq 6$; it is *much less*, in a sense that we will explain in Subsection 2.4. For now, we simply state that while $S_n(1234)$ is "roughly" 9^n, the value of $S_n(1342)$ is"roughly" 8^n. Second, the formula is, in some sense, simpler than that for $S_n(1234)$. Indeed, it follows from Theorem 3 that the ordinary generating function of the sequence $S_n(1342)$ is

$$H(x) = \sum_{i \geq 0} F^i(x) = \frac{1}{1 - F(x)} = \frac{32x}{-8x^2 + 20x + 1 - (1-8x)^{3/2}}.$$

This is an *algebraic* power series. On the other hand, it is known (Problem Plus 5.10 in [7] that the ordinary generating function of the sequence $S_n(1234)$ is *not* algebraic. So permutations avoiding the monotone pattern are not even the *nicest* among permutations avoiding a given pattern, in terms of the generating functions that count them.

There is no known formula for the third sequence, that of the numbers $S_n(1324)$. However, the following inequality is known [6].

Theorem 4 *For all integers $n \geq 7$, the inequality*

$$S_n(1234) < S_n(1324)$$

holds.

Proof: Let us call an entry of a permutation a *right-to-left maximum* if it is larger than all entries on its right. Then let us say that two n-permutations are in the same class if they have the same left-to-right minima, and they are in the same positions, and they have the same right-to-left maxima, and they are in the same positions as well. For example, 51234 and 51324 are in the same class, but $z = 24315$ and $v = 24135$ are not, as the third entry of z is not a left-to-right minimum, whereas that of v is.

It is straightforward to see that each non-empty class contains exactly one 1234-avoiding permutation, the one in which the subsequence of entries that are neither left-to-right minima nor right-to-left maxima is decreasing.

It is less obvious that each class contains *at least one* 1324-avoiding permutation. Note that if a permutation contains a 1324-pattern, then we can choose such a pattern so that its first element is a left-to-right minimum and its last element is a right-to-left maximum. Take a 1324-avoiding permutation, and take one of its 1324-patterns of the kind described in the previous sentence. Interchange its second and third element. Observe that this will keep the permutation within its original class. Repeat this procedure as long as possible. The procedure will stop after a finite number of steps since each step decreases the number of inversions of the permutation. When the procedure stops, the permutation at hand avoids 1324.

This shows that $S_n(1234) \leq S_n(1324)$ for all n. If $n \geq 7$, then the equality cannot hold since there is at least one class that contains more than one 1324-avoiding permutation. For $n = 7$, this is the class $3 * 1 * 7 * 5$, which contains 3612745 and 3416725. For larger n, this class can be prepended by $n(n-1) \cdots 8$ to get a suitable class. $\diamond$

It turns out again that $S_n(1324)$ is *much* larger than $S_n(1234)$. We will give the details in Subsection 2.4.

2.3 Patterns of Any Length

For general k, there are some good estimates known for the value of $S_n(\alpha_k)$. The first one can be proved by an elementary method.

Theorem 5 *For all positive integers n and $k > 2$, we have*

$$S_n(123\cdots k) \leq (k-1)^{2n}.$$

Proof: Let us say that an entry x of a permutation is of rank i if it is the end of an increasing subsequence of length i, but there is no increasing subsequence of length $i+1$ that ends in x. Then for all i, elements of rank i must form a decreasing subsequence. Therefore, a q-avoiding permutation can be decomposed into the union of $k-1$ decreasing subsequences. Clearly, there are at most $(k-1)^n$ ways to partition our n entries into $k-1$ blocks. Then we have to place these blocks of entries somewhere in our permutation. There are at most $(k-1)^n$ ways to assign each position of the permutation to one of these blocks, completing the proof. $\diamond$

Indeed, Theorem 5 has a stronger version, obtained by Amitaj Regev [23]. It needs heavy analytic machinery, and therefore will not be proved here. We mention the result, however, as it shows that no matter what k is, the constant $(k-1)^2$ in Theorem 5 cannot be replaced by a smaller number, so the elementary estimate of Theorem 5 is optimal in some strong sense. We remind the reader that functions $f(n)$ and $g(n)$ are said to be *asymptotically equal* if $\lim_{n\rightarrow\infty}\frac{f(n)}{g(n)} = 1$.

Theorem 6 *[23] For all n, $S_n(1234\cdots k)$ asymptotically equals*

$$\lambda_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}.$$

Here

$$\lambda_k = \gamma_k^2 \int\limits_{x_1\geq x_2\geq\cdots\geq x_k}\int\cdots\int [D(x_1,x_2,\cdots,x_k)\cdot e^{-(k/2)x^2}]^2 dx_1 dx_2\cdots dx_k,$$

where $D(x_1,x_2,\cdots,x_k) = \Pi_{i<j}(x_i - x_j)$, and $\gamma_k = (1/\sqrt{2\pi})^{k-1}\cdot k^{k^2/2}$.

2.4 Stanley-Wilf Limits

The following celebrated result of Adam Marcus and Gábor Tardos [21] shows that in general, it is very difficult to avoid any given pattern q.

Theorem 7 *[21] For all patterns q, there exists a constant c_q so that*

$$S_n(q) \leq c_q^n. \tag{4}$$

It is not difficult [2] to show using Fekete's lemma that the sequence $(S_n(q))^{1/n}$ is monotone increasing. The previous theorem shows that it is bounded from above, leading to the following.

Corollary 1 *For all patterns q, the limit*

$$L(q) = \lim_{n\to\infty} (S_n(q))^{1/n}$$

exists.

The real number $L(q)$ is called the *Stanley-Wilf limit*, or *growth rate* of the pattern q. In this terminology, Theorem 6 implies that $L(\alpha_k) = (k-1)^2$. In particular, $L(1234) = 9$, while Theorem 3 implies that $L(1342) = 8$. So it is not simply easier to avoid 1234 than 1342, it is *exponentially* easier to do so.

Numerical evidence suggests that in the multiset of $k!$ real numbers $S_n(q)$, the numbers $S_n(\alpha_k)$ are much closer to the maximum than to the minimum. This led to the plausible conjecture that for any pattern q of length k, the inequality $L(q) \leq (k-1)^2$ holds. This would mean that while there are patterns of length k that are easier to avoid than α_k, there are none that are much easier to avoid, in the sense of Stanley-Wilf limits. However, this conjecture has been disproved by the following result of Michael Albert and al.

Theorem 8 *[1] The inequality $L(1324) \geq 11.35$ holds.*

In other words, it is not simply harder to avoid 1234 than 1324, it is *exponentially* harder to do so.

2.5 Asymptotic Normality

In this section we change direction and prove that the distribution of the number of copies of α_k in a randomly selected n-permutation converges in distribution to a normal distribution. (For the rest of this paper, when we say random permutation of length n, we always assume that each n-permutation is selected with probability $1/n!$.) Note that in the special case of $k = 2$, this is equivalent to the classic result that the distribution of inversions in random permutations is asymptotically normal. See [14] and its references for various proofs of that result, or [11] for a generalization.

We need to introduce some notation for transforms of the random variable Z. Let $\bar{Z} = Z - E(Z)$, let $\tilde{Z} = \bar{Z}/\sqrt{\mathrm{Var}(Z)}$, and let $Z_n \to N(0,1)$ mean that Z_n converges in distribution to the standard normal variable.

Our main tool in this section will be a theorem of Svante Janson [19]. In order to be able to state that theorem, we need the following definition.

Definition 1 *Let $\{Y_{n,k} | k = 1, 2, \cdots, N_n\}$ be an array of random variables. We say that a graph G is a* dependency graph *for $\{Y_{n,k} | k = 1, 2 \cdots, N_n\}$ if the following two conditions are satisfied:*

1. *There exists a bijection between the random variables $Y_{n,k}$ and the vertices of G, and*
2. *If V_1 and V_2 are two disjoint sets of vertices of G so that no edge of G has one endpoint in V_1 and another one in V_2, then the corresponding sets of random variables are independent.*

Note that the dependency graph of a family of variables is not unique. Indeed if G is a dependency graph for a family and G is not a complete graph, then we can get other dependency graphs for the family by simply adding new edges to G.

Now we are in position to state Janson's theorem, the famous *Janson dependency criterion.*

Theorem 9 *[19] Let $Y_{n,k}$ be an array of random variables such that for all n, and for all $k = 1, 2, \cdots, N_n$, the inequality $|Y_{n,k}| \leq A_n$ holds for some real number A_n, and that the maximum degree of a dependency graph of $\{Y_{n,k} | k = 1, 2, \cdots, N_n\}$ is Δ_n.*

Set $Y_n = \sum_{k=1}^{N_n} Y_{n,k}$ and $\sigma_n^2 = Var(Y_n)$. If there is a natural number m so that

$$N_n \Delta_n^{m-1} \left(\frac{A_n}{\sigma_n} \right)^m \to 0, \tag{5}$$

as n goes to infinity, then

$$\tilde{Y}_n \to N(0, 1).$$

Let us order the $\binom{n}{k}$ subsequences of length k of the permutation $p_1 p_2 \cdots p_n$ linearly in some way. For $1 \leq i \leq \binom{n}{k}$, let $X_{n,i}$ be the indicator random variable of the event that in a randomly selected permutation of length n, the ith subsequence of length k in the permutation $p = p_1 p_2 \cdots p_n$ is a $12 \cdots k$-pattern. We will now verify that the family of the $X_{n,i}$ satisfies all conditions of the Janson Dependency Criterion.

First, $|X_{n,i}| \leq 1$ for all i and all n, since the $X_{n,i}$ are indicator random variables. So we can set $A_n = 1$. Second, $N_n = \binom{n}{k}$, the total number of subsequences of length k in p. Third, if $a \neq b$, then $X_{n,a}$ and $X_{n,b}$ are independent unless the corresponding subsequences intersect. For that, the bth subsequence must intersect the ath subsequence in j entries, for some $1 \leq j \leq k-1$. For a fixed ath subsequence, the number of ways that can happen is $\sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} = \binom{n}{k} - \binom{n-k}{k} - 1$, where we used the well-known Vandermonde identity to compute the sum. Therefore,

$$\Delta_n \leq \binom{n}{k} - \binom{n-k}{k} - 1. \tag{6}$$

In particular, note that (6) provides an upper bound for Δ_n in terms of a polynomial function of n that is of degree $k-1$ since terms of degree k will cancel.

There remains the task of finding a lower bound for σ_n that we can then use in applying Theorem 9. Let $X_n = \sum_{i=1}^{\binom{n}{k}} X_{n,i}$. We will show the following.

Proposition 1 *There exists a positive constant c so that for all n, the inequality*

$$Var(X_n) \geq cn^{2k-1}$$

holds.

Proof: By linearity of expectation, we have

$$\begin{aligned}
\mathrm{Var}(X_n) &= E(X_n^2) - (E(X_n))^2 && (7)\\
&= E\left(\left(\sum_{i=1}^{\binom{n}{k}} X_{n,i}\right)^2\right) - \left(E\left(\sum_{i=1}^{\binom{n}{k}} X_{n,i}\right)\right)^2 && (8)\\
&= E\left(\left(\sum_{i=1}^{\binom{n}{k}} X_{n,i}\right)^2\right) - \left(\sum_{i=1}^{\binom{n}{k}} E(X_{n,i})\right)^2 && (9)\\
&= \sum_{i_1,i_2} E(X_{n,i_1}X_{n,i_2}) - \sum_{i_1,i_2} E(X_{n,i_1})E(X_{n,i_2}). && (10)
\end{aligned}$$

Let I_1 (resp. I_2) denote the k-element subsequence of p indexed by i_1, (resp. i_2). Clearly, it suffices to show that

$$\sum_{|I_1 \cap I_2| \leq 1} E(X_{n,i_1}X_{n,i_2}) - \sum_{i_1,i_2} E(X_{n,i_1})E(X_{n,i_2}) \geq cn^{2k-1}, \tag{11}$$

since the left-hand side of (11) is obtained from the (10) by removing the sum of some positive terms, that is, the sum of all $E(X_{n,i_1}X_{n,i_2})$ where $|I_1 \cap I_2| > 1$.

As $E(X_{n,i}) = 1/k!$ for each i, the sum with negative sign in (10) is

$$\sum_{i_1,i_2} E(X_{n,i_1})E(X_{n,i_2}) = \binom{n}{k}^2 \cdot \frac{1}{k!^2},$$

which is a polynomial function in n, of degree $2k$ and of leading coefficient $\frac{1}{k!^4}$. As far as the summands in (10) with a positive sign go, *most* of them are also equal to $\frac{1}{k!^2}$. More precisely, $E(X_{n,i_1}X_{n,i_2}) = \frac{1}{k!^2}$ when I_1 and I_2 are disjoint, and that happens for $\binom{n}{k}\binom{n-k}{k}$ ordered pairs (i_1, i_2) of indices. The sum of these summands is

$$d_n = \binom{n}{k}\binom{n-k}{k}\frac{1}{k!^2}, \tag{12}$$

which is again a polynomial function in n, of degree $2k$ and with leading coefficient $\frac{1}{k!^4}$. So summands of degree $2k$ will cancel out in (10). (We will see in the next paragraph that the summands we have not yet considered add up to a polynomial of degree $2k-1$.) In fact, considering the two types of summands we studied in (10) and (12), we see that they add up to

$$\binom{n}{k}\binom{n-k}{k}\frac{1}{k!^2} - \binom{n}{k}^2\frac{1}{k!^2} \quad = \quad n^{2k-1}\frac{2\binom{k}{2} - \binom{2k-1}{2}}{k!^4} + O(n^{2k-2}) \tag{13}$$

$$= \quad n^{2k-1}\frac{-k^2}{k!^4} + O(n^{2k-2}). \tag{14}$$

Next we look at ordered pairs of indices (i_1, i_2) so that the corresponding subsequences I_1 and I_2 intersect in exactly one entry, the entry x. Let us say that counting from the left, x is the ath entry in I_1, and the bth entry in I_2. See Figure 1 for an illustration.

Observe that $X_{n,i_1}X_{n,i_2} = 1$ if and only if all of the following independent events occur.

(a) In the $(2k-1)$-element set of entries that belong to $I_1 \cup I_2$, the entry x is the $(a+b-1)$th smallest. This happens with probability $1/(2k-1)$.

(b) The $a+b-2$ entries on the left of x in $I_1 \cup I_2$ are all smaller than the $2k-a-b$ entries on the right of x in $I_1 \cup I_2$. This happens with probability $\frac{1}{\binom{2k-2}{a+b-2}}$.

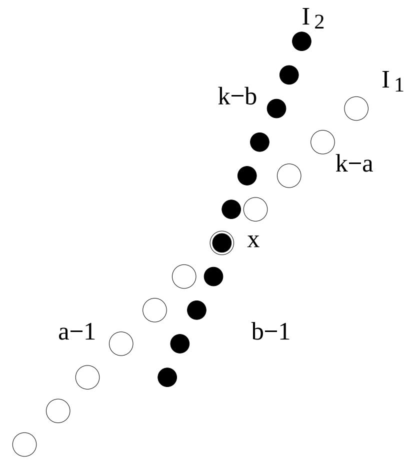

Figure 1: In this example, $k = 11$, $a = 7$, and $b = 5$.

(c) The subsequences of I_1 on the left of x and on the right of x, and the subsequences of I_2 on the left of x and on the right of x are all monotone increasing. This happens with probability $\frac{1}{(a-1)!(b-1)!(k-a)!(k-b)!}$.

Therefore, if $|I_1 \cap I_2| = 1$, then

$$P(X_{i_1}X_{i_2} = 1) = \frac{1}{(2k-1)\binom{2k-2}{a+b-2}(a-1)!(b-1)!(k-a)!(k-b)!} \tag{15}$$

$$= \frac{1}{(2k-1)!} \cdot \binom{a+b-2}{a-1}\binom{2k-a-b}{k-a}. \tag{16}$$

How many such ordered pairs (I_1, I_2) are there? There are $\binom{n}{2k-1}$ choices for the underlying set $I_1 \cup I_2$. Once that choice is made, the $a+b-1$st smallest entry of $I_1 \cup I_2$ will be x. Then the number of choices for the set of entries other than x that will be part of I_1 is $\binom{a+b-2}{a-1}\binom{2k-a-b}{k-a}$. Therefore, summing over all a and b and recalling (15),

$$p_n = \sum_{|I_1 \cap I_2|=1} P(X_{i_1}X_{i_2} = 1) = \sum_{|I_1 \cap I_2|=1} E(X_{i_1}X_{i_2}) \tag{17}$$

$$= \frac{1}{(2k-1)!}\binom{n}{2k-1}\sum_{a,b}\binom{a+b-2}{a-1}^2\binom{2k-a-b}{k-a}^2. \tag{18}$$

The expression we just obtained is a polynomial of degree $2k-1$, in the variable n. We claim that its leading coefficient is larger than $k^2/k!^4$. If we can show that, the proposition will be proved since (14) shows that the summands not included in (17) contribute about $-\frac{k^2}{k!^4}n^{2k-1}$ to the left-hand side of (11).

Recall that by the Cauchy-Schwarz inequality, if $t_1, t_2, \cdots, t_m$ are non-negative real numbers, then

$$\frac{\left(\sum_{i=1}^m t_i\right)^2}{m} \leq \sum_{i=1}^m t_i^2, \tag{19}$$

where equality holds if and only if all the t_i are equal.

Let us apply this inequality with the numbers $\binom{a+b-2}{a-1}^2\binom{2k-a-b}{k-a}^2$ playing the role of the t_i, where a and b range from 1 to k. We get that

$$\sum_{1\leq a,b\leq k}\binom{a+b-2}{a-1}^2\binom{2k-a-b}{k-a}^2 > \frac{\left(\sum_{1\leq a,b\leq k}\binom{a+b-2}{a-1}\binom{2k-a-b}{k-a}\right)^2}{k^2}. \tag{20}$$

We will use Vandermonde's identity to compute the right-hand side. To that end, we first compute the sum of summands with a *fixed* $h = a+b$. We obtain

$$\begin{aligned}\sum_{1\leq a,b\leq k}\binom{a+b-2}{a-1}\binom{2k-a-b}{k-a} &= \sum_{h=2}^{2k}\sum_{a=1}^{k}\binom{h-2}{a-1}\binom{2k-h}{k-a} && (21)\\ &= \sum_{h=2}^{2k}\binom{2k-2}{k-1} && (22)\\ &= (2k-1)\cdot\binom{2k-2}{k-1}. && (23)\end{aligned}$$

Substituting the last expression into the right-hand side of (20) yields

$$\sum_{1\leq a,b\leq k}\binom{a+b-2}{a-1}^2\binom{2k-a-b}{k-a}^2 > \frac{1}{k^2}\cdot(2k-1)^2\cdot\binom{2k-2}{k-1}^2. \tag{24}$$

Therefore, (17) and (24) imply that

$$p_n > \frac{1}{(2k-1)!}\binom{n}{2k-1}\frac{(2k-1)^2}{k^2}\binom{2k-2}{k-1}^2.$$

As we pointed out after (17), p_n is a polynomial of degree $2k-1$ in the variable n. The last displayed inequality shows that its leading coefficient is larger than

$$\frac{1}{(2k-1)!^2}\cdot\frac{1}{k^2}\cdot\frac{(2k-2)!^2}{(k-1)!^4} = \frac{k^2}{k!^4}$$

as claimed.

Comparing this with (14) completes the proof of our Proposition. $\diamond$

We can now return to the application of Theorem 9 to our variables $X_{n,i}$. By Proposition 1, there is an absolute constant C so that $\sigma_n > Cn^{k-0.5}$ for all n. So (5) will be satisfied if we show that there exists a positive integer m so that

$$\binom{n}{k}(dn^{k-1})^{m-1} \cdot (n^{-k+0.5})^m < dn^{-0.5m} \to 0.$$

Clearly, any positive integer m is a good choice. So we have proved the following theorem.

Theorem 10 *Let k be a fixed positive integer, and let X_n be the random variable counting occurrences of α_k in permutations of length n. Then $\tilde{X}_n \to N(0,1)$. In other words, X_n is asymptotically normal.*

3 Monotone Subsequences with Entries in Consecutive Positions

In 2001, Sergi Elizalde and Marc Noy [12] considered similar problems using another definition of pattern containment. Let us say that the permutation $p = p_1p_2\cdots p_n$ *tightly* contains the permutation $q = q_1q_2\cdots q_k$ if there exists an index $0 \leq i \leq n-k$ so that $q_j < q_r$ if and only if $p_{i+j} < p_{i+r}$. (We point out that this definition is a very special case of the one introduced by Babson and Steingrímsson in [3] and called *generalized pattern avoidance*, but we will not need that much more general concept in this paper.)

Example 2 *While permutation 246351 contains 132 (take the second, third, and fifth entries), it does not* tightly contain *132 since there are no three entries in consecutive positions in 246351 that would form a 132-pattern.*

If p does not tightly contain q, then we say that p *tightly avoids* q. Let $T_n(q)$ denote the number of n-permutations that tightly avoid q. An intriguing conjecture of Elizalde and Noy [12] is the following.

Conjecture 1 *For any pattern q of length k and for any positive integer n, the inequality*

$$T_n(q) \leq T_n(\alpha_k)$$

holds.

This is in stark contrast with the situation for traditional patterns, where, as we have seen in the previous section, the monotone pattern is not the easiest or the hardest to avoid, even in the sense of growth rates.

3.1 Tight Patterns of Length Three

Conjecture 1 is proved in [12] in the special case of $k = 3$. As it is clear by taking reverses and complements that $T_n(123) = T_n(321)$ and that $T_n(132) = T_n(231) = T_n(213) = T_n(312)$, it suffices to show that $T_n(132) < T_n(123)$ if $n \geq n$. The authors achieve that by a simple injection.

It turns out that the numbers $T_n(123)$ are not simply larger than the numbers $T_n(132)$; they are larger even in the sense of logarithmic asymptotics. The following results contain the details.

Theorem 11 *[12] Let $A_{123}(x) = \sum_{n \geq 0} T_n(123) \frac{x^n}{n!}$ be the exponential generating function of the sequence $\{T_n(123)\}_{n \geq 0}$. Then*

$$A_{123}(x) = \frac{\sqrt{3}}{2} \cdot \frac{e^{x/2}}{\cos\left(\frac{\sqrt{3}}{2}x + \frac{\pi}{6}\right)}.$$

Furthermore,

$$T_n(123) \sim \gamma_1 \cdot (\rho_1)^n \cdot n!,$$

where $\rho_1 = \frac{3\sqrt{3}}{2\pi}$ and $\gamma_1 = e^{3\sqrt{3}\pi}$.

Theorem 12 *[12] Let $A_{132}(x) = \sum_{n \geq 0} T_n(132) \frac{x^n}{n!}$ be the exponential generating function of the sequence $\{T_n(132)\}_{n \geq 0}$. Then*

$$A_{132}(x) = \frac{1}{1 - \int_0^x e^{-t^2/2} dt}.$$

Furthermore,

$$T_n(132) \sim \gamma_2 \cdot (\rho_2)^n \cdot n!,$$

where ρ_2^{-1} is the unique positive root of the equation $\int_0^x e^{-t^2/2} dt = 1$, and $\gamma_2 = e^{(\rho_2)^{-2}/2}$.

3.2 Tight Patterns of Length Four

For tight patterns, the case of length four is even more complex than it is for traditional patterns. Indeed, it is not true that each of the 24 sequences $T_n(q)$, where q is a tight pattern of length four, is identical to one of $T_n(1342)$,

$T_n(1234)$, and $T_n(1324)$. In fact, in [12], Elizalde and Noy showed that there are exactly seven distinct sequences of this kind. They have also proved the following results.

Theorem 13 *We have*

1. $T_n(1342) \sim \gamma_1(\rho_1)^n \cdot n!$,
2. $T_n(1234) \sim \gamma_2(\rho_2)^n \cdot n!$, *and*
3. $T_n(1243) \sim \gamma_3(\rho_3)^n \cdot n!$,

where ρ_1^{-1} *is the smallest positive root* z *of the equation* $\int_0^z = e^{-t^3/6}dt = 1$, ρ_2^{-1} *is the smallest positive root of* $\cos z - \sin z + e^{-z} = 0$, *and* ρ_3 *is the solution of a certain equation involving Airy functions.*

The approximate values of these constants are

- $\rho_1 = 0.954611$, $\gamma_1 = 1.8305194$,
- $\rho_2 = 0.963005$, $\gamma_2 = 2.2558142$,
- $\rho_3 = 0.952891$, $\gamma_3 = 1.6043282$.

These results are interesting for several reasons. First, we see that again, $T_n(\alpha_4)$ is larger than the other $T_n(q)$, even in the asymptotic sense. Second, $T_n(1234) \neq T_n(1243)$, in contrast to the traditional case, where $S_n(1234) = S_n(1243)$. Third, the tight pattern 1342 is *not* the hardest to avoid, unlike in the traditional case, where $S_n(1342) \leq S_n(q)$ for any pattern q of length four.

3.3 Longer Tight Patterns

For tight patterns that are longer than four, the only known results concern monotone patterns. They have been found by Richard Warlimont, and, independently, also by Sergi Elizalde and Marc Noy.

Theorem 14 *[12], [28], [29] For all integers* $k \geq 3$, *the identity*

$$\sum_{n\geq 0} T_n(\alpha_k)\frac{x^n}{n!} = \left(\sum_{i\geq 0}\frac{x^{ik}}{(ik)!} - \sum_{i\geq 0}\frac{x^{ik+1}}{(ik+1)!}\right)^{-1}$$

holds.

Theorem 15 *[29] Let $k \geq 3$, let $f_k(x) = \sum_{i\geq 0} \frac{x^{ik}}{(ik)!} - \sum_{i\geq 0} \frac{x^{ik+1}}{(ik+1)!}$, and let ω_k denote the smallest positive root of $f_k(x)$. Then*

$$\omega_k = 1 + \frac{1}{m!}\left(1 + O(1)\right),$$

and

$$\frac{T_n(\alpha_k)}{n!} \sim c_m \omega_k^{-n}.$$

3.4 Growth Rates

The form of the results in Theorems 11 and 12 is not an accident. They are special cases of the following general theorem.

Theorem 16 *[13] For all patterns q, there exists a constant w_q so that*

$$\lim_{n\to\infty} \left(\frac{T_n(q)}{n!}\right)^{1/n} = w_q.$$

Compare this with the result of Corollary 1. That Corollary and the fact that the sequence $(S_n(q))^{1/n}$ is increasing, show that the numbers $S_n(q)$ are roughly as large as $L(q)^n$, for some constant $L(q)$. Clearly, it is much easier to avoid a tight pattern than a traditional pattern. However, Theorem 16 shows how much easier it is. Indeed, this time it is not the *number* of pattern avoiding permutations is simply exponential; it is their *ratio* to all permutations that is exponential.

The fact that $T_n(q)/n! < C_q^n$ for *some* C_q is straightforward. Indeed, $T_n(q)/n! < \left(\frac{k!-1}{k!}\right)^{\lfloor n/k \rfloor}$ by simply looking at $\lfloor n/k \rfloor$ distinct subsequences of k consecutive entries. Interestingly, Theorem 16 shows that this straightforward estimate is optimal in some (weak) sense. Note that there is no known way to get a result similarly close to the truth for traditional patterns.

3.5 Asymptotic Normality

Our goal now is to prove that the distribution of tight copies of α_k is asymptotically normal in randomly selected permutations of length n. Note that in the special case of $k = 2$, our problem is reduced to the classic result stating that descents of permutations are asymptotically normal. (Just as in the previous section, see [14] and its references for various proofs of this fact, or [11] for a generalization.) Our method is very similar to the one we used in Subsection 2.5. For fixed n and $1 \leq i \leq n - k + 1$, let $Y_{n,i}$

denote the indicator random variable of the event that in $p = p_1 p_2 \cdots p_n$, the subsequence $p_i p_{i+1} \cdots p_{i+k-1}$ is increasing. Set $Y_n = \sum_{i=1}^{n-k+1} Y_{n,i}$. We want to use Theorem 9. Clearly, $|Y_{n,i}| \leq 1$ for every i, and $N_n = n - k + 1$. Furthermore, the graph with vertex set $\{1, 2, \cdots, n-k+1\}$ in which there is an edge between i and j if and only if $|i-j| \leq k-1$ is a dependency graph for the family $\{Y_{n,i} | 1 \leq i \leq n-k+1\}$. In this graph, $\Delta_n = 2k-2$. We will prove the following estimate for $\mathrm{Var}(Y)$.

Proposition 2 *There exists a positive constant C so that $Var(Y) \geq cn$ for all n.*

Proof: By linearity of expectation, we have

$$\begin{aligned} \mathrm{Var}(Y_n) &= E(Y_n^2) - (E(Y_n))^2 && (25) \\ &= E\left(\left(\sum_{i=1}^{n-k+1} Y_{n,i}\right)^2\right) - \left(E\left(\sum_{i=1}^{n-k+1} Y_{n,i}\right)\right)^2 && (26) \\ &= E\left(\left(\sum_{i=1}^{n-k+1} Y_{n,i}\right)^2\right) - \left(\sum_{i=1}^{n-k+1} E(Y_{n,i})\right)^2 && (27) \\ &= \sum_{i_1,i_2} E(Y_{n,i_1} Y_{n,i_2}) - \sum_{i_1,i_2} E(Y_{n,i_1}) E(Y_{n,i_2}). && (28) \end{aligned}$$

In (28), all the $(n-k+1)^2$ summands with a negative sign are equal to $1/k!^2$. Among the summands with a positive sign, the $(n-2k+1)(n-2k+2)$ summands in which $|i_1 - i_2| \geq k$ are equal to $1/k!^2$, the $n-k+1$ summands in which $i_1 = i_2$ are equal to $1/k!$, and the $2(n-2k+2)$ summands in which $|i_1 - i_2| = k-1$ are equal to $1/(k+1)!$. All remaining summands are non-negative. This shows that

$$\begin{aligned} \mathrm{Var}(Y_n) &\geq \frac{n(1-2k) + 3k^2 - 2k + 1}{k!^2} + \frac{n-k+1}{k!} + \frac{2(n-k+2)}{(k+1)!} \\ &\geq \left(\frac{1}{k!} + \frac{2}{(k+1)!} - \frac{2k-1}{k!^2}\right) n + d_k, \end{aligned}$$

where d_k is a constant that depends only on k. As the coefficient $\frac{1}{k!} + \frac{2}{(k+1)!} - \frac{2k-1}{k!^2}$ of n in the last expression is positive for all $k \geq 2$, our claim is proved. $\diamond$

The main theorem of this subsection is now immediate.

Theorem 17 *Let Y_n denote the random variable counting tight copies of α_k in a randomly selected permutation of length n. Then $\tilde{Y}_n \to N(0,1)$.*

Proof: Use Theorem 9 with $m = 3$, and let C be the constant of Proposition 2. Then (5) simplifies to

$$(n-k+1) \cdot (2k-2)^2 \cdot \frac{C^3}{n^{1.5}},$$

which converges to 0 as n goes to infinity. $\diamond$

4 Consecutive Entries in Consecutive Positions

Let us take the idea of Elizalde and Noy one step further, by restricting the notion of pattern containment further as follows. Let $p = p_1 p_2 \cdots p_n$ be a permutation, let $k < n$, and let $q = q_1 q_2 \cdots q_k$ be another permutation. We say that p *very tightly* contains q if there is an index $0 \leq i \leq n-k$ and an integer $0 \leq a \leq n-k$ so that $q_j < q_r$ if and only if $p_{i+j} < p_{i+r}$, and,

$$\{p_{i+1}, p_{i+2}, \cdots, p_{i+k}\} = \{a+1, a+2, \cdots, a+k\}.$$

That is, p very tightly contains q if p tightly contains q and the entries of p that form a copy of q are not just in consecutive positions, but they are also consecutive as integers (in the sense that their set is an interval). We point out that this definition was used by A. Myers [22] who called it *rigid* pattern avoidance. However, in order to keep continuity with our previous definitions, we will refer to it as very tight pattern avoidance.

For example, 15324 tightly contains 132 (consider the first three entries), but does not very tightly contain 132. On the other hand, 15324 very tightly contains 213, as can be seen by considering the last three entries. If p does not very tightly contain q, then we will say that *p very tightly avoids q*.

4.1 Enumerative Results

Let $V_n(q)$ be the number of permutations of length n that very tightly avoid the pattern q. The following early results on $V_n(\alpha_k)$ are due to David Jackson and others. They generalize earlier work by Riordan [24] concerning the special case of $k = 3$.

Theorem 18 *[18], [17] For all positive integers n, and any $k \leq n$, the value of $V_n(\alpha_k)$ is equal to the coefficient of x^n in the formal power series*

$$\sum_{m \geq 0} m! x^m \left(\frac{1 - x^{k-1}}{1 - x^k} \right)^m .$$

Note that in particular, this implies that for $k \leq n < 2k$, the number of permutations of length $k+r$ *containing* a very tight copy of α_k is $r!(r^2+r+1)$.

4.2 An Extremal Property of the Monotone Pattern

Recall that we have seen in Section 2 that in the multiset of the $k!$ numbers $S_n(q)$ where q is of length k, the number $S_n(\alpha_k)$ is neither minimal nor maximal. Also recall that in Section 3 we mentioned that in the multiset of the $k!$ numbers $T_n(q)$, where q is of length k, the number $T_n(\alpha_k)$ is *conjectured* to be maximal. While we cannot prove that we prove that in the multiset of the $k!$ numbers $V_n(q)$, where q is of length k, the number $V_n(\alpha_k)$ is maximal, in this Subsection we prove that for almost all very tight patterns q of length k, the inequality $V_n(q) \leq V_n(\alpha_k)$ does hold.

4.2.1 An Argument Using Expectations

Let q be any pattern of length k. For a fixed positive integer n, let $X_{n,q}$ be the random variable counting the very tight copies of q in a randomly selected n-permutation. It is straightforward to see that by linearity of expectation,

$$E(X_{n,q}) = \frac{(n-k+1)^2}{\binom{n}{k} k!}. \tag{29}$$

In particular, $E(X_{n,q})$ does not depend on q, just on the length k of q.

Let $p_{n,i,q}$ be the probability that a randomly selected n-permutation contains *exactly* i very tight copies of q, and let $P(n,i,q)$ be the probability that a randomly selected n-permutation contains *at least* i very tight copies of q. Note that $V_n(q) = (1 - P(n,1,q))n!$, for any given pattern q.

Now note that by the definition of expectation

$$
\begin{aligned}
E(X_n, q) &= \sum_{i=1}^{m} i p_{n,i,q} \\
&= \sum_{j=0}^{m-1} \sum_{i=0}^{j} p_{n,m-i,q} \\
&= p_{n,m,q} + (p_{n,m,q} + p_{n,m-1,q}) + \cdots + (p_{n,m,q} + \cdots + p_{n,1,q}) \\
&= \sum_{i=1}^{m} P(n,i,q).
\end{aligned}
$$

We know from (29) that $E(X_{n,q}) = E(X_{n,\alpha_k})$, and then the previous displayed equation implies that

$$\sum_{i=1}^{m} P(n,i,q) = \sum_{i=1}^{m} P(n,i,\alpha). \tag{30}$$

So if we can show that for $i \geq 2$, the inequality

$$P(n,i,q) \leq P(n,i,\alpha_k) \tag{31}$$

holds, then (30) will imply that $P(n,1,q) \geq P(n,1,\alpha_k)$, which is equivalent to $V_n(q) \leq V_n(\alpha_k)$, which we set out to prove.

4.2.2 Extendible and Non-extendible Patterns

Now we are going to describe the set of patterns q for which we will prove that $V_n(q) \leq V_n(\alpha_k)$.

Let us assume that the permutation $p = p_1 p_2 \cdots p_n$ very tightly contains two *non-disjoint* copies of the pattern $q = q_1 q_2 \cdots q_k$. Let these two copies be $q^{(1)}$ and $q^{(2)}$, so that $q^{(1)} = p_{i+1} p_{i+2} \cdots p_{i+k}$ and $q^{(2)} = p_{i+j+1} p_{i+j+2} \cdots p_{i+j+k}$ for some $j \in [1, k-1]$. Then $|q^{(1)} \cap q^{(2)}| = k - j + 1 =: s$. Furthermore, since the set of entries of $q^{(1)}$ is an interval, and the set of entries of $q^{(2)}$ is an interval, it follows that the set of entries of $q^{(1)} \cap q^{(2)}$ is also an interval. So the rightmost s entries of q, and the leftmost s entries of q must form identical patterns, and the respective sets of these entries must both be intervals.

If q' is the reverse of the pattern q, then clearly $V_n(q) = V_n(q')$. Therefore, we can assume without loss of generality that that the first entry of q is less than the last entry of q. For shortness, we will call such patterns *rising* patterns.

We claim that if p very tightly contains two non-disjoint copies $q^{(1)}$ and $q^{(2)}$ of the rising pattern q, and s is defined as above, then the *rightmost* s entries of q must also be the *largest* s entries of q. This can be seen by considering $q^{(1)}$. Indeed, the set of these entries of $q^{(1)}$ is the intersection of two intervals of the same length, and therefore, must be an ending segment of the interval that starts on the left of the other. An analogous argument, applied for $q^{(2)}$, shows that the leftmost s entries of q must also be the *smallest* s entries of q. So we have proved the following.

Proposition 3 *Let p be a permutation that very tightly contains copies $q^{(1)}$ and $q^{(2)}$ of the pattern $q = q_1 q_2 \cdots q_k$. Let us assume without loss of generality that q is rising. Then $q^{(1)}$ and $q^{(2)}$ are disjoint unless all of the following hold.*

There exists a positive integer $s \leq k-1$ so that

1. *the rightmost s entries of q are also the largest s entries of q, and the leftmost s entries of q are also the smallest s entries of q, and*
2. *the pattern of the leftmost s entries of q is identical to the pattern of the rightmost s entries of q.*

If q satisfies both of these criteria, then two very tightly contained copies of q in p may indeed intersect. For example, the pattern $q = 2143$ satisfies both of the above criteria with $s = 2$, and indeed, 214365 very tightly contains two intersecting copies of q, namely 2143 and 4365.

The following definition is similar to one in [22].

Definition 2 *Let $q = q_1 q_2 \cdots q_k$ be a rising pattern that satisfies both conditions of Proposition 3. Then we say that q is* extendible.

If q is rising and not extendible, then we say that q is non-extendible.

Note that the notions of extendible and non-extendible patterns are only defined for rising patterns here.

Example 3 *The extendible patterns of length four are as follows:*

- *1234, 1324 (here $s = 1$),*
- *2143 (here $s = 2$).*

Now we are in a position to prove the main result of this Subsection.

Theorem 19 *Let q be any pattern of length k so that either q or its reverse q' is non-extendible. Then for all positive integers n,*

$$V_n(q) \leq V_n(\alpha_k).$$

Proof: We have seen in Subsubsection 4.2.1 that it suffices to prove (31).

On the one hand,

$$\frac{(n-k-i+2)!}{n!} \leq P(n, i, \alpha_k), \tag{32}$$

since the number of n-permutations very tightly containing i copies of α is at least as large as the number of n-permutations very tightly containing the pattern $12 \cdots (i+k-1)$. The latter is at least as large as the number of n-permutations that very tightly contain a $12 \cdots (i+k-1)$-pattern in their first $i+k-1$ positions.

On the other hand,

$$P(n, i, q) \leq \binom{n-i(k-1)}{i}^2 (n-ik)! \frac{1}{n!}. \tag{33}$$

This can be proved by noting that if S is the i-element set of starting positions of i (necessarily disjoint) very tight copies of q in an n-permutation, and A_S is the event that in a random permutation $p = p_1 \cdots p_n$, the subsequence $p_j p_{j+1} \cdots p_{j+k-1}$ is a very tight q-subsequence for all $j \in S$, then $P(A_S) = \binom{n-i(k-1)}{i}(n-ik)!\frac{1}{n!}$. The details can be found in [10].

Comparing (32) and (33), the claim of the theorem follows. Again, the reader is invited to consult [10] for details. $\diamond$

It is not difficult to show [10] that the ratio of extendible permutations of length k among all permutations of length k converges to 0 as k goes to infinity. So Theorem 19 covers almost all patterns of length k.

4.3 The Limiting Distribution of the Number of Very Tight Copies

In the previous two sections, we have seen that the limiting distribution of the number of copies of α_k, as well as the limiting distribution of the number of tight copies of α_k, is normal. Very tight copies behave differently. We will discuss the special case of $k = 2$, that is, the case of the very tight pattern 12.

Theorem 20 *Let Z_n be the random variable that counts very tight copies of 12 in a randomly selected permutation of length n. Then Z_n converges a Poisson distribution with parameter $\lambda = 1$.*

A version of this result was proved, in a slightly different setup, by Wolfowitz in [31] and by Kaplansky in [20]. They used the *method of moments*, which is the following.

Lemma 1 *[25] Let U be a random variable so that*

1. *for every positive integer k, the moment $E(U^k)$ exists, and*
2. *the variable U is completely determined by its moments, that is, there is no other variable with the same sequence of moments.*

Let $U_1, U_2, \cdots$ be a sequence of random variables, and let us assume that for all positive integers k,

$$\lim_{n\to\infty} U_n^k = U^k.$$

Then $U_n \to U$ in distribution.

Proof: (of Theorem 20.) It is well-known [27] that the Poisson distribution (with any parameter) is determined by its moments, so the method of moments can be applied to prove convergence to a Poisson distribution. Let $Z_{n,i}$ be the indicator random variable of the event that in a randomly selected n-permutation $p = p_1p_2\cdots p_n$, the inequality $p_i + 1 = p_{i+1}$. Then $E(Z_{n,i}) = 1/n$, and the probability that p has a very tight copy of α_k for $k > 2$ is $O(1/n)$. Therefore, we have

$$\lim_{n\to\infty} E(Z_n^j) = \lim_{n\to\infty} E\left(\left(\sum_{i=1}^{n-1} Z_{n,i}\right)^j\right) = \lim_{n\to\infty} E\left(\left(\sum_{i=1}^{n-1} V_{n,i}\right)^j\right), \quad (34)$$

where the $V_{n,i}$ are *independent* random variables and each of them takes value 0 with probability $(n-1)/n$, and value 1 with probability $1/n$. (See [31] for more details.) The rightmost limit in the above displayed equation is not difficult to compute. Let t be a fixed non-negative integer. Then the probability that exactly t variables $V_{n,i}$ take value 1 is $\binom{n-1}{t}n^{-t}(\frac{n-1}{n})^{n-t} \sim \frac{e^{-1}}{t!}$. Once we know the t-element set of the $V_{n,i}$ that take value 1, each of the t^j strings of length j formed from those t variables contributes 1 to $E(V^j)$. Summing over all t, this proves that

$$\lim_{n\to\infty} E\left(\left(\sum_{i=1}^{n-1} V_{n,i}\right)^j\right) = e^{-1}\sum_{t\geq 0}\frac{t^j}{j!}.$$

On the other hand, it is well-known that $e^{-1}\sum_{t\geq 1}\frac{t^j}{j!}$, the jth Bell number, is also the jth moment of the Poisson distribution with parameter 1. Comparing this to (34), we see that the sequence $E(Z_n^j)$ converges to the jth moment of the Poisson distribution with parameter 1. Therefore, by the method of moments, our claim is proved. $\diamond$

5 Added In Proof

While this is a survey on monotone patterns, it is worth pointing out that Theorem 10, and its proof, survive even if we replace α_k by an *arbitrary* pattern. Most of the proof carries through without modification. All that has to be changed are the independent events (b) and (c) considered following equation (14).

Recall that we are in the special case when I_1 and I_2 both form q-patterns, and $I_1 \cap I_2 = x$ is the ath smallest entry in I_1 and the bth smallest entry in I_2. Given q, the pair (a,b) describes the location of x in I_1 and in I_2 as well. Let I_1' (resp. I_2') denote the set of $a-1$ positions in I_1 (resp. $b-1$ positions in I_2) which must contain entries smaller than x given that I_1 (resp. I_2) forms a q-pattern. Similarly, let I_1'' (resp. I_2'') denote the set of $k-a$ positions in I_1 (resp. $k-b$ positions I_2) which must contain entries larger than x given that I_1 (resp. I_2) forms a q-pattern.

Now leave condition (a) unchanged, and change conditions (b) and (c) as follows.

(b') The $a+b-2$ entries in positions belonging to $I_1' \cup I_2'$ must all be smaller than the $2k-a-b$ entries in positions belonging to $I_1'' \cup I_2''$. This happens with probability $\frac{1}{\binom{2k-2}{a+b-2}}$.

(c')
- the subsequence I_1' is a pattern that is isomorphic to the pattern formed by the $a-1$ smallest entries of q,
- the subsequence I_2' is a pattern that is isomorphic to the pattern formed by the $b-1$ smallest entries of q,
- the subsequence I_1'' is a pattern that is isomorphic to the pattern formed by the $k-a$ largest entries of q, and
- the subsequence I_2'' is a pattern that is isomorphic to the pattern formed by the $k-b$ largest entries of q.

This happens with probability $\frac{1}{(a-1)!(b-1)!(k-a)!(k-b)!}$.

The rest of the proof is unchanged. It is worth pointing out that while the expectation of the number of copies of a pattern of a given length k does not depend on the pattern (it is $\binom{n}{k}/k!$), the variance of these numbers does. However, it follows easily from our work that the variance is a polynomial function of n that has degree $2n-1$, and that the *leading coefficient* of this polynomial does not depend on q. It is the terms of lower degree that depend on q.

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