

Real Zeros and Normal Distribution for statistics on Stirling permutations defined by Gessel and Stanley

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Abstract

We study Stirling permutations defined by Gessel and Stanley in [8]. We prove that their generating function according to the number of descents has real roots only. We use that fact to prove that the distribution of these descents, and other, equidistributed statistics on these objects converge to a normal distribution.

1 Introduction

In [8] Ira Gessel and Richard Stanley defined an interesting class of multiset permutations called *Stirling Permutations*. Let Q_n denote the set of all permutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ in which for all i , all entries between the two occurrences of i are larger than i . For instance, Q_2 has three elements, namely 1122, 1221, and 2211. It is not difficult to see that Q_n has $1 \cdot 3 \cdot \dots \cdot (2n-1) = (2n-1)!!$ elements. Gessel and Stanley then proved many enumerative results for these permutations and showed several connections between these and other combinatorial objects, such as set partitions.

Counting Stirling permutations by descents, the authors of [8] found a recurrence relation similar to the recurrence relation known for classic permutations. In this paper, we will continue in that direction. First, we show the simple but interesting fact that on Q_n the descent and the *plateau* statistics, to be defined in the next section, are equidistributed. Then we prove that for any fixed n , the generating polynomial of all Stirling permutations

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in Q_n with respect to the descent statistic has real roots only. This is analogous to the well-known case (see Theorem 1.33 of [1]) of classic permutations, namely the result that all the roots of Eulerian polynomials are real. Finally, we apply a classic result of Bender to use this real roots property to prove that the descents of Stirling permutations in Q_n are normally distributed.

2 Stirling Permutations and Real Zeros

Let $q = a_1 a_2 \cdots a_{2n} \in Q_n$ be a Stirling permutation. Let the index i be called an *ascent* of q if $i = 0$ or $a_i < a_{i+1}$, let i be called a *descent* of q if $i = 2n$ or $a_i > a_{i+1}$, and let i be called a *plateau* of q if $a_i = a_{i+1}$. It is obvious that the ascent and descent statistics are equidistributed, since reversing an element of Q_n turns ascents into descents and vice versa. It is somewhat less obvious that the plateau statistic is also equidistributed with the previous two. This fact, and a reason for it, are the content of the next proposition. Note that its first identity, (1), was proved in [8].

Proposition 1 *Let $C_{n,i}$ be the number of elements of Q_n with i descents. Then for all positive integers $n, i \geq 2$, we have*

$$C_{n,i} = iC_{n-1,i} + (2n - i)C_{n-1,i-1}. \quad (1)$$

Similarly, let $c_{n,i}$ be number of elements of Q_n with i plateaux. Then for all positive integers $n, i \geq 2$, we have

$$c_{n,i} = ic_{n-1,i} + (2n - i)c_{n-1,i-1}. \quad (2)$$

In particular, since $C_{1,1} = c_{1,1} = 1$ and $C_{1,0} = c_{1,0} = 0$, the identity

$$C_{n,i} = c_{n,i} \quad (3)$$

holds.

Proof: There are two ways to obtain an element of Q_n from an element $p \in Q_{n-1}$ by inserting two copies of n into consecutive positions. Either p must have i descents, and then we insert the two copies of n into a descent, or p has $i - 1$ descents, and then we insert the two consecutive copies of n into one of the $(2n - 1) - (i - 1) = 2n - i$ positions that are not descents.

The argument proving (2) is analogous. \diamond

Corollary 1 *On average, elements of Q_n have $(2n+1)/3$ ascents, $(2n+1)/3$ descents, and $(2n+1)/3$ plateaux.*

Proposition 1 enables us to prove a strong result on the roots of the polynomials $\sum_{i=1}^n C_{n,i}x^i$. The method we use follows an idea of H. Wilf ([9], [1] Theorem 1.33) who used it on classic permutations.

Theorem 1 *Let $C_n(x) = \sum_{i=1}^n C_{n,i}x^i$. Then for all positive integers n , the roots of the polynomial $C_n(x)$ are all real, distinct, and non-positive.*

Proof: For $n = 1$, one sees that $C_1(x) = x$, and the statement holds. For $n = 2$, one sees that $C_2(x) = 2x^2 + x = x(2x + 1)$, and so the statement again holds.

For $n \geq 3$, recurrence relation (1) implies

$$C_n(x) = (x - x^2)C'_{n-1}(x) + (2n - 1)x C_{n-1}(x) \quad (4)$$

as can be seen by equating coefficients of x^i . The right-hand side is similar to the derivative of a product, which suggests the following rearrangement

$$C_n(x) = x(1 - x)^{2n} \frac{d}{dx} ((1 - x)^{1-2n} C_{n-1}(x)). \quad (5)$$

Let us now assume inductively that the roots of $C_{n-1}(x)$ are real, distinct and non-positive. Clearly, $C_n(x)$ vanishes at $x = 0$. Furthermore, by Rolle's theorem, (5) shows that $C_n(x)$ has a root between any pair of consecutive roots of $C_{n-1}(x)$. This counts for $n - 1$ roots of $C_n(x)$. So the last root must also be real, since complex roots of polynomials with real coefficients must come in conjugate pairs.

There remains to show that the last root of $C_n(x)$ must be on the right of the rightmost root of C_{n-1} . Consider (4) at the rightmost root x_0 of C_{n-1} . As x_0 is negative, we know that $x_0 - x_0^2 < 0$, and so $C_n(x_0)$ and $C'_{n-1}(x_0)$ have opposite signs. The claim now follows, since in $-\infty$, the polynomials $C_n(x)$ and $C'_{n-1}(x)$ must converge to the same (infinite) limit as their degrees are of the same parity. As $C'_{n-1}(x)$ has no more roots on the right of x_0 , the polynomial $C_n(x)$ must have one. \diamond

Note that we have in fact proved that the roots of $C_{n-1}(x)$ and $C_n(x)$ are interlacing, so the sequence C_1, C_2, \dots is a *Sturm sequence*.

As an immediate application of the real zeros property, we can determine where peak (or peaks) of the sequence $C_{n,1}, C_{n,2}, \dots, C_{n,n}$ is. Our tool in doing so is the following theorem of Darroch.

Theorem 2 [4] *Let $A(x) = \sum_{k=0}^n a_k x^k$ be a polynomial that has real roots only that satisfies $A(1) > 0$. Let m be an index so that $a_m = \max_{0 \leq i \leq n} a_i$. Let $\mu = A'(1)/A(1)$. Then we have*

$$|\mu - m| < 1.$$

In particular, a sequence with the real zeros property can have at most two peaks. Note that $A'(1) = \sum_{i=0}^n i a_i$ and $A(1) = \sum_{i=0}^n a_i$, therefore $A'(1)/A(1)$ is nothing else but the weighted average of the coefficients a_i , with i being the weight of a_i . So in the particular case when $A(x) = C_n(x)$, we have

$$\begin{aligned} \frac{C'_n(1)}{C_n(1)} &= \frac{\sum_i i C_{n,i}}{\sum_i C_{n,i}} \\ &= \sum_i i \cdot \frac{C_{n,i}}{(2n-1)!!} \\ &= \frac{2n+1}{3}, \end{aligned}$$

where the last step follows from Corollary 1. Indeed, $\frac{C_{n,i}}{(2n-1)!!}$ is just the probability that a randomly selected Stirling permutation of length n has exactly i descents, so $\sum_i i \cdot \frac{C_{n,i}}{(2n-1)!!}$ is just the expected number of descents in such permutations.

Therefore, by Theorem 2, we obtain the following result.

Theorem 3 *Let i be an index so that $C_{n,i} = \max_k C_{n,k}$. Then*

1. $i = (2n+1)/3$ if $(2n+1)/3$ is an integer, and
2. $i = \lfloor (2n+1)/3 \rfloor$ or $i = \lceil (2n+1)/3 \rceil$ if $(2n+1)/3$ is not an integer.

3 Stirling Permutations and Normal Distribution

In this section, we prove that the plateaux (equivalently ascents, equivalently, descents) of Stirling permutations are normally distributed. Our main tool is the following result of Bender. Let X_n be a random variable, and let $a_n(k)$ be a triangular array of non-negative real numbers, $n = 1, 2, \dots$, and $1 \leq k \leq m(n)$ so that

$$P(X_n = k) = p_n(k) = \frac{a_n(k)}{\sum_{i=1}^{m(n)} a_n(i)}.$$

Set $g_n(x) = \sum_{k=1}^{m(n)} p_n(k)x^k$.

We need to introduce some notation for transforms of the random variable Z . Let $\bar{Z} = Z - E(Z)$, let $\tilde{Z} = \bar{Z}/\sqrt{\text{Var}(Z)}$, and let $Z_n \rightarrow N(0, 1)$ mean that Z_n converges in distribution to the standard normal variable.

Theorem 4 [2] *Let X_n and $g_n(x)$ be as above. If $g_n(x)$ has real roots only, and*

$$\sigma_n = \sqrt{\text{Var}(X_n)} \rightarrow \infty,$$

then $\tilde{X}_n \rightarrow N(0, 1)$.

See [3] for related results.

We want to use Theorem 4 to prove that the plateaux of permutations in Q_n are normally distributed. Because of Theorem 1, all we need for that is to prove that the variance of the number of these plateaux converges to infinity as n goes to infinity. We will accomplish more by proving an explicit formula for this variance. In order to state that formula, let $Y_{n,i}$ be the indicator random variable of the event that in a randomly selected element of Q_n , the two copies of i are consecutive, that is, they form a plateau. Note that $P(Y_{n,n} = 1) = E(Y_{n,n}) = 1$. Set $Y_n = \sum_{i=1}^n Y_{n,i}$.

Theorem 5 *For all positive integers n , the equality*

$$\text{Var}(Y_n) = \frac{2n^2 - 2}{18n - 9} \tag{6}$$

holds.

Proof: We are going to use the identity $\text{Var}(Y_n) = E(Y_n^2) - E(Y_n)^2$. We have seen in Corollary 1 that $E(Y_n) = \frac{2n+1}{3}$. Let $s_n = E(Y_n^2)$. The key element of our computations is the following lemma.

Lemma 1 *For all positive integers n , the equality*

$$s_{n+1} = \frac{2n-1}{2n+1} \cdot s_n + \frac{4n+4}{3}. \tag{7}$$

holds.

Proof: In order to prove (7), we need the following simple facts.

Proposition 2 1. For all positive integers n , and all indices $i \neq j$ that satisfy $1 \leq i, j \leq n$, the equality

$$E(Y_{n+1,i}Y_{n+1,j}) = \frac{2n-1}{2n+1}E(Y_{n,i}Y_{n,j})$$

holds.

2. For all positive integers n and all indices $1 \leq i \leq n$, the equality

$$E(Y_{n+1,i}) = \frac{2n}{2n+1}E(Y_{n,i})$$

holds.

3. For all indices $i \leq n+1$, the equality

$$E(Y_{n+1,i}Y_{n+1,n+1}) = E(Y_{n+1,i})$$

holds. In particular, $E(Y_{n+1,n+1}) = 1$.

Proof:

1. In order to get an element of Q_{n+1} in which i and j are both plateaux, take an element of Q_n in which i and j are both plateaux, and insert two consecutive copies of $n+1$ into any of the $2n-1$ available places, that is, anywhere but between the two copies of i or the two copies of j .
2. In order to get an element of Q_{n+1} in which i is a plateau, insert two consecutive copies of $n+1$ into any of the $2n$ available slots, that is, anywhere but between the two copies of i .
3. Obvious since $n+1$ is always a plateau in elements of Q_{n+1} .

◇

We return to proving Lemma 1.

Note that $s_{n+1} = \sum_{1 \leq i, j \leq n+1} E(Y_{n+1,i}Y_{n+1,j})$. The latter can be split into partial sums based on whether i or j are equal to $n+1$ as follows.

$$\begin{aligned} s_{n+1} = & \sum_{1 \leq j \leq n+1} E(Y_{n+1,n+1}Y_{n+1,j}) + \sum_{1 \leq i \leq n} E(Y_{n+1,i}Y_{n+1,n+1}) \\ & + \sum_{1 \leq i, j \leq n} E(Y_{n+1,i}Y_{n+1,j}). \end{aligned}$$

Based on part 3 of Proposition 2, this simplifies to

$$s_{n+1} = \sum_{1 \leq j \leq n+1} E(Y_{n+1,j}) + \sum_{1 \leq i \leq n} E(Y_{n+1,i}) + \sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} E(Y_{n+1,i}Y_{n+1,j}) \\ + \sum_{1 \leq i \leq n} E(Y_{n+1,i}).$$

Now note that the first sum on the right-hand side is just $E(Y_{n+1})$, the second sum is $E(Y_{n+1} - Y_{n+1,n+1}) = E(Y_{n+1}) - 1$, use part 1 of Proposition 2 on the third sum, and part 2 of Proposition 2 on the fourth sum to get

$$s_{n+1} = 2E(Y_n) - 1 + \frac{2n-1}{2n+1}(s_n - E(Y_n)) + \frac{2n}{2n+1}E(Y_n).$$

Recalling from Corollary 1 that $E(Y_n) = \frac{2n+1}{3}$, this reduces to (7). \diamond

Using the recursive formula proved in Lemma 1, it is routine to prove that

$$s_n = E(Y_n^2) = \frac{8n^3 + 6n^2 - 2n - 3}{18n - 9}. \quad (8)$$

Therefore, $\text{Var}(Y_n) = s_n - E(Y_n)^2 = \frac{2n^2-2}{18n-9}$ as claimed. \diamond

Theorem 6 *The distribution of the number of plateaux of elements of Q_n converges to a normal distribution as n goes to infinity. That is, $\tilde{Y}_n \rightarrow N(0,1)$.*

Proof: Let $X_n = Y_n$, and let $g_n(x) = \frac{1}{(2n-1)!!}C_n(x)$. Then Theorem 1 and Theorem 5 show that the conditions of Theorem 4 are satisfied, and the claim follows from Theorem 4. \diamond

4 Remarks

Corollary 1 shows that $E(Y_n) = (2n+1)/3$. It is not difficult to prove that $E(Y_{n,n-i}) = \prod_{j=1}^i \frac{2n-2j}{2n-2j+1}$. By the linearity of expectation this proves the interesting identity

$$\sum_{i=0}^{n-1} \prod_{j=1}^i \frac{2n-2j}{2n-2j+1} = \frac{2n+1}{3},$$

where the empty product (indexed by $i = 0$) is considered to be 1.

The proof of the equidistribution of the descent and plateau statistics we gave is very simple, but it is of recursive nature. It can be used to define an algorithm that recursively constructs a bijection f from the set of permutations in Q_n that have k descents into the set of permutations in Q_n that have k plateaux. Let us assume that such a bijection has already been constructed for Q_{n-1} , and any $k \leq n - 1$. If $p \in Q_n$, and p has k descents, then let $p' \in Q_{n-1}$ be the permutation obtained from p by removing the two copies of n . Let q' be the image of p' under the bijection already constructed for Q_{n-1} . If p is obtained from p' by inserting two copies of n into the i th descent of p' , then let $f(p) = q$ be the permutation obtained from q' by inserting the two copies of n into the i th plateau of q . If p is obtained from p' by inserting two copies of n into the j th non-descent of p' , then let $f(p) = q$ be the permutation obtained from q' by inserting the two copies of n into the j th non-descent of q' .

A direct bijective proof has recently been given by Hyeong-Kwan Ju [7].

We mention that the results of this work have recently been extended by Janson [5], and Janson et al. [6].

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