

STACK WORDS AND A BOUND FOR 3-STACK SORTABLE PERMUTATIONS

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ABSTRACT. We use stack words to find a new, simple proof for the best known upper bound for the number of 3-stack sortable permutations of a given length. This is the first time that stack words are used to obtain such a result.

1. INTRODUCTION

Let $p = p_1 p_2 \cdots p_n$ be a permutation. In order to stack sort p , we consider its entries one by one. First, we take p_1 , and put it in a one-dimensional sorting device, *the stack*. The stack stands vertically, and is open at the top; it can hold entries that are increasing from the top of the stack to the bottom of the stack. Second, we take p_2 . If $p_2 < p_1$, then it is allowed for p_2 to go in the stack on top of p_1 , so we put p_2 there. If $p_2 > p_1$, however, then first we take p_1 out of the stack, and put it to the first position of the output permutation, and *then* we put p_2 into the stack. We continue this way: at step i , we compare p_i with the entry r currently on the top of the stack. If $p_i < r$, then p_i goes on the top of the stack; if not, then r goes to the next empty position of the output permutation, and p_i gets compared to the new entry that is now at the top of the stack. The algorithm ends when all n entries passed through the stack and are in the output permutation $s(p)$. See Section 8.2 of [2] for a survey on stack sortable permutations.

Definition 1.1. *If the output permutation $s(p)$ defined by the above algorithm is the identity permutation $123 \cdots n$, then we say that p is stack sortable.*

It is well-known that a permutation p is stack sortable if and only if it avoids the pattern 231, hence the number of stack sortable permutations of length n is the Catalan number $C_n = \binom{2n}{n}/(n+1)$.

A permutation is called t -stack sortable if sending it through the stack t times results in the identity permutation. In other words, p is t -stack sortable if $s^t(p)$ is the identity permutation. Enumerating t -stack sortable permutations for $t > 1$ is extremely difficult. Let $W_t(n)$ be the number of t -stack sortable permutations of length n . For $t = 2$, the following formula was conjectured by West in [9]. The formula turned out to be quite difficult

to prove, but now has several complicated proofs. See [10], [7], [6] or [5] for various proofs.

Theorem 1.2. *For all positive integers n , the number of 2-stack sortable permutations of length n is given by*

$$W_2(n) = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}.$$

There is no formula known for $W_t(n)$ if $t \geq 3$. Even good upper bounds are difficult to find. An easy upper bound is given by the inequality $W_t(n) \leq (t+1)^{2n}$, which follows from the fact that if p is t -stack sortable, then p avoids the pattern $23 \cdots n1$. Characterizing t -stack sortable permutations is very complicated if $t > 2$, though the case of $t = 3$ has been done by Claesson and Úlfarsson [3], and by Úlfarsson [8]. For the case of general t , a less concrete description was recently obtained by Albert, Bouvel and Féray in [1].

Colin Defant [4] proved upper bounds for $\lim_{n \rightarrow \infty} \sqrt[t]{W_t(n)}$ for the cases of $t = 3$ and $t = 4$ that are better than the $(t+1)^2$ bound mentioned above. For $t = 3$, he proved the upper bound $\lim_{n \rightarrow \infty} \sqrt[3]{W_3(n)} \leq 12.53296$, and for $t = 4$, he proved the upper bound $\lim_{n \rightarrow \infty} \sqrt[4]{W_4(n)} \leq 21.97225$. Results on related lower bounds can be found in Defant's paper [5], where it is shown that $\lim_{n \rightarrow \infty} \sqrt[3]{W_3(n)} \geq 8.659702$ and that $\lim_{n \rightarrow \infty} \sqrt[t]{W_t(n)} \geq (\sqrt{t} + 1)^2$, along with a new proof for the formula for $W_2(n)$, and a polynomial time algorithm to compute the numbers $W_3(n)$.

In this paper, we provide a new, simpler proof for Defant's upper bound for the case of $t = 3$. Another feature of our proof is that it uses stack words, that have been long known to describe t -stack sortable permutations, but never used to prove an upper bound for them.

2. OUR METHOD

Consider the following modification of the t -stack sorting operation. Instead of passing a permutation through a stack t times, we pass it through t stacks placed next to each other in series as follows. The first stack operates as the usual stack except that when an entry x leaves it, it does not go to the output right away. It goes to the next stack if $x < j$, where j is the entry on the top of the next stack, or if the next stack is empty. If $j < x$, then x cannot move until j does.

The general step of this algorithm is as follows. Let S_1, S_2, \dots, S_t be the t stacks, with a_i being the entry on top of stack S_i . If the next entry x of the input is smaller than a_1 , we put x on top of S_1 . Otherwise, we find the smallest i so that a_i can move to the next stack (that is, that $a_i < a_{i+1}$ or S_{i+1} is empty), and move a_i on top of S_{i+1} . If we do not find such i , or if S_1, S_2, \dots, S_{t-1} and the input have all been emptied out, then we put the entry on the top of S_t into the output.

We can describe the movement of the entries of the input permutation p through the stack by *stack words*. If $t = 1$, then there are just two kinds of

moves, an entry either moves in the stack or out of the stack. Let us denote these steps with letters A and B , respectively. Then the movement of all entries of p is described by a stack word consisting of n copies of A and n copies of B in which for all i , the i th A precedes the i th B . The number of such words is well-known to be $C_n = \binom{2n}{n}/(n+1)$. On the other hand, if p is stack sortable, then its output is the identity, so given the stack word of p , we can uniquely recover p . It is easy to prove by strong induction that each word that satisfies the conditions described in this paragraph is indeed the stack word of a stack sortable permutation, so this is a stack word proof of the fact that $W_1(n) = C_n$.

In general, if there are t stacks, then there are $t+1$ different kinds of moves. Therefore, the movement of p through the $t+1$ stacks can be described by a word of length $(t+1)n$ that consists of n copies of each of $t+1$ different letters.

In particular, if $t = 2$, then there are three kind of moves, and if $t = 3$, the case that is the subject of our paper, then there are four kinds of moves, which we will treat as follows. Let A denote the move of an entry from the input to the first stack, let B denote the move of an entry from the first stack to the second stack, let C denote the move of an entry from the second stack to the third stack, and let D denote the move of an entry from the third stack to the output.

Note that we will also call stack words 3-stack words or 2-stack words when we want to emphasize the number of stacks that are used to sort a given word. Also note, for future reference, that for all t , the descents of p are in bijective correspondence with the AA factors of p . (An XY factor of a word is just a letter X immediately followed by a letter Y .)

We will identify 3-stack sortable permutations with their stack words. We can do that since if p is 3-stack sortable, then its image under the 3-stack sorting algorithm is the identity permutation, so given the stack word of p , we can uniquely recover p .

Proposition 2.1. *Let w be a 3-stack word of a permutation. Then all of the following hold.*

- (1) *There is no BB factor in w .*
- (2) *There is no CC factor in w .*
- (3) *There is no BAB factor in w .*
- (4) *There is no CBA^jC factor in w , where $j \geq 0$.*

Proof. Each of these statements holds because otherwise the entries in the second or third stack would not be increasing from the top of the stack to the bottom of the stack. \square

Proposition 2.2. *Let w be a 3-stack word of a permutation. Then all of the following hold.*

- (1) *There is no DA factor in w .*
- (2) *There is no DB factor in w .*

(3) *There is no CA factor in w .*

Proof. Each of these statements holds because of the greediness of our algorithm. For instance, a D cannot be followed by an A , since the move corresponding to D did not change the content of the first stack, so if the A move was possible after the D move, it was possible before the D move, and therefore, *it would have been made* before the D move. Analogous considerations imply the other two statements. \square

Note that the conditions given in Propositions 2.1 and 2.2 are necessary, that is, they must hold in 3-stack words of all permutations, but they are *not* sufficient. In other words, if a word satisfies all these conditions, it is not necessarily the 3-stack word of a permutation.

Let w be a 3-stack word of a 3-stack sortable permutation p , and let $v = v(w)$ be the subword of w that consists of the letters B , C and D in w . In other words, $v = v(w)$ is the word obtained from w by removing all copies of the letter A . This can create BB factors in v , even though there were no BB factors in w .

Note that v describes how the stack sorted image $s(p)$ of p traverses the second and third stacks. Note that as p is 3-stack sortable, $s(p)$ is 2-stack sortable. So v is the 2-stack word of the 2-stack sortable permutation $s(p)$ over the alphabet $\{B, C, D\}$. Therefore, there are $W_2(n) = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}$ possible choices for v .

Furthermore, every descent of $s(p)$ bijectively corresponds to a BB -factor of v . The number of 2-stack sortable permutations of length n with $k - 1$ descents is known (see Problem Plus 8.1 in the book [2]) to be

$$(1) \quad W_2(n, k - 1) = \frac{(n + k - 1)!(2n - k)!}{k!(n + 1 - k)!(2k - 1)!(2n - 2k + 1)!}.$$

3. COMPUTING THE UPPER BOUND

Lemma 3.1. *The number $W_3(n)$ of 3-stack sortable permutations of length n satisfies the inequality*

$$W_3(n) \leq \sum_{k=1}^{(n+1)/2} \frac{(n + k - 1)!(2n - k)!}{k!(n + 1 - k)!(2k - 1)!(2n - 2k + 1)!} \cdot \binom{2n - 2k}{n - 1}.$$

Proof. Let us count all such permutations with respect to the number of descents of their stack sorted image $s(p)$. If $s(p)$ has $k - 1$ descents, then its 2-stack word v has $k - 1$ factors BB . In order to recover the 3-stack word w of p , we must insert n copies of A into v so that we get a valid 3-stack word. As BB factors and BAB factors are not allowed in w , we must insert two copies of A into the middle of every BB factor, and we also have to put one A in front of the first B . We have $n - 2(k - 1) - 1 = n - 2k + 1$ copies of A left. We can insert these only in n possible slots, namely on the left of the first B , and immediately following any B except the last one. (This is because Proposition 2.2 tells us that there are no CA or DA factors in

w.) Therefore, by a classic balls-and-boxes argument, the number of ways to place all copies of A is at most

$$\binom{n - 2k + 1 + n - 1}{n - 1} = \binom{2n - 2k}{n - 1}.$$

As there are $W_2(n, k - 1)$ choices for v , the proof is complete by summing over all possible values of k . \square

Theorem 3.2. *The inequality*

$$\lim_{n \rightarrow \infty} \sqrt[n]{W_3(n)} \leq 12.53296$$

holds.

Proof. As Lemma 3.1 provides an upper bound for $W_3(n)$ as a sum of less than n summands, it suffices to prove that the largest of those summands is of exponential order 12.539547. In order to do that, we use Stirling's formula that states that $m! \sim (m/e)^m \sqrt{2\pi m}$, so $\lim_{m \rightarrow \infty} \sqrt[m]{m!} = m/e$. Let $w_3(n, k)$ denote the number of 3-stack sortable permutations p of length n so that $s(p)$ has k descents. Setting $k = nx$, with $x \in (0, 1]$, and applying Stirling's formula to each factor of the bound in Lemma 3.1, this leads to the equality

$$\begin{aligned} g(x) : &= \lim_{n \rightarrow \infty} \sqrt[n]{w_3(n, xn)} \\ &= \frac{(1+x)^{1+x} \cdot (2-x)^{2-x} \cdot (2-2x)^{2-2x}}{x^x \cdot (1-x)^{1-x} \cdot (2x)^{2x} \cdot (2-2x)^{2-2x} \cdot (1-2x)^{1-2x}} \\ &= (1+x) \cdot (2-x)^{2-x} \cdot x^{-3x} \cdot (1-x)^{-1+x} \cdot (1-2x)^{2x-1} \cdot \left(\frac{x+1}{4}\right)^x. \end{aligned}$$

The function g takes its maximum when $g'(x) = 0$, which occurs when

$$x = \frac{1}{12} \cdot (27 + 12 \cdot \sqrt{417})^{1/3} - \frac{13}{4 \cdot (27 + 12 \cdot \sqrt{417})^{1/3}} + \frac{1}{4} \approx 0.2883918927.$$

For that value of x , we get $g(x) = 12.53296$, completing the proof. \square

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