

On Two Related Questions of Wilf Concerning Standard Young Tableaux

Miklós Bóna
Department of Mathematics
University of Florida
Gainesville FL 32611-8105
USA

November 5, 2008

Abstract

We consider two questions of Wilf related to Standard Young Tableaux. We provide a partial answer to one question, and that will lead us to a more general answer to the other question. Our answers are purely combinatorial.

1 Introduction

In 1992, in his paper [9], Herb Wilf has proved the following interesting result.

Theorem 1 (*Wilf, [9].*) *Let $u_k(n)$ be the number of permutations of length n that contain no increasing subsequence of length $k + 1$, and let $y_k(n)$ be the number of Standard Young Tableaux on n boxes that have no rows longer than k . Then for all even positive integers k , the equality*

$$\binom{2n}{n} u_k(n) = \sum_{r=0}^{2n} \binom{2n}{r} (-1)^r y_k(r) y_k(2n - r) \quad (1)$$

holds.

Wilf's proof of Theorem 1 was not elementary; it used modified Bessel functions and computed the determinant of a Toeplitz matrix. Therefore, Wilf asked the following two intriguing questions.

1. Is there a purely combinatorial proof for Theorem 1 ?
2. What statement corresponds to Theorem 1 for odd k ?

In this paper, we answer Question 1 in a special case, which then will lead us to a more general answer to Question 2. This answer will be a formula that will still contain a summation sign, but each summand will be non-negative, which will explain why the answer is always non-negative. The number of nonzero summands will be half of what it is in (1), and the summands will be significantly smaller than in (1).

We point out that in another special case, that of $k = 2$, a simple and elegant bijective proof has recently been given by Rebecca Smith and Micah Coleman [2].

We will assume familiarity with the Robinson-Schensted correspondence between permutations of length n and pairs of Standard Young Tableaux on n boxes and of the same shape. In particular, we will need the following facts.

1. There is a one-to-one correspondence RS between involutions on an n -element set and Standard Young Tableaux on n boxes.
2. The length of the longest increasing subsequence of the involution v is equal to the length of the first row of $RS(v)$, and
3. the length of the longest decreasing subsequence of the involution v is equal to the length of the first column of $RS(v)$.

Readers who want to deepen their knowledge of the Robinson-Schensted correspondence should consult the book [5] of Bruce Sagan. The Robinson-Schensted correspondence makes Theorem 1 even more intriguing, since both sides of (1) can be interpreted in terms of Standard Young Tableaux as well as in terms of permutations.

In Section 3, we will also need the following, somewhat less well-known result of Janet Simpson Beissinger, which is implicit in an earlier paper of Marcel-Paul Schützenberger [6].

Theorem 2 [1] *Let v be an involution, and let $RS(v)$ be its image under the Robinson-Schensted correspondence. Then the number of fixed points of v is equal to the number of odd columns of $RS(v)$.*

2 When $k = 2n$

In this section, we bijectively prove Theorem 1 in the special case when $k = 2n$. It is clear that in that special case, the requirement on the increasing subsequences on the left-hand side of (1), and the requirement on the length of rows on the right-hand side of (1) are automatically satisfied. Therefore, if $y(m)$ denotes the number of involutions of an m -element set, then Theorem 1 simplifies to the following proposition.

Proposition 1 *For all positive integers n , we have*

$$\binom{2n}{n} n! = \sum_{r=0}^{2n} \binom{2n}{r} (-1)^r y(r) y(2n - r). \quad (2)$$

Proof: Let $[i]$ denote the set $\{1, 2, \dots, i\}$. Let A_n be the set of all permutations of the elements of n -element subsets of $[2n]$. Then the left-hand side of (2) is equal to $|A_n|$.

Let B_n be the set of ordered pairs (p, q) , where p is an involution on a subset s_p of $[2n]$, and q is an involution on the set $[2n] - s_p$, the complement of s_p in $[2n]$. Then the right-hand side of (2) counts the elements of B_n taking the parity of r into account. More precisely, the right-hand side of (2) is equal to the number of elements of B_n in which $|s_p|$ has an even size minus the number of elements of B_n in which $|s_p|$ has an odd size.

Now we are going to define an involution f on a subset of B_n . Let $(p, q) \in B_n$. As p is an involution, all cycles of p are of length one (these are also called fixed points) or length two. Let $F(p, q)$ be the set of all fixed points of p and of all fixed points of q . Let $M(p, q)$ be the maximal element of $F(p, q)$ as long as $F(p, q)$ is a non-empty set. Now *move* $M(p, q)$ to the *other* involution in (p, q) . That is, if $M(p, q)$ was a fixed point of p , then move $M(p, q)$ to q , and if $M(p, q)$ was a fixed point of q , then move M to p . Call the resulting pair of involutions $f(p, q) = (p', q')$.

Example 1 *Let $n = 4$, let $p = (31)(62)(5)$, and let $q = (7)(84)$. Then $F(p, q) = \{5, 7\}$, so $M(p, q) = 7$, and therefore, $f(p, q) = (p', q')$, where $p' = (31)(5)(62)(7)$ and $q' = (84)$.*

It is clear that $f(p', q') = (p, q)$, since $F(p, q) = F(p', q')$, and so $M(p, q) = M(p', q')$. So applying f a second time simply moves $M(p, q)$ back to its original place.

As the number of elements in p and in p' differs by exactly one, these two numbers are of different parity, and so the total contribution of (p, q)

and $f(p, q)$ to the right-hand side of (2) is 0. Therefore, the only pairs (p, q) whose contribution is not canceled by the contribution of $f(p, q)$ are the pairs for which $f(p, q)$ is *not defined*, that is, pairs (p, q) in which both p and q are *fixed point-free involutions*.

Noting that fixed point-free involutions are necessarily of even length, this shows that (2) will be proved if we can show that

$$\binom{2n}{n} n! = \sum_{r=0}^{2n} \binom{2n}{r} x(r)x(2n-r), \quad (3)$$

where $x(r)$ is the number of *fixed point-free involutions* of length r .

This equality is straightforward to prove computationally, using the fact that $x(2t) = (2t-1) \cdot (2t-3) \cdots 1 = (2t-1)!!$ and $x(2t+1) = 0$. However, for the sake of combinatorial purity, we provide a bijective proof.

The left-hand side counts the ways to choose n elements a_1, a_2, \dots, a_n of $[2n]$ and then to arrange them in a line. Let $a \in A_n$ denote such a choice and arrangement. Now let $i_1 < i_2 < \dots < i_n$ be the elements of $[2n]$ that we did *not* choose, listed increasingly. Take the fixed point-free involution whose cycles are the 2-cycles (i_j, a_j) , for $1 \leq j \leq n$. Color the cycles in which $i_j < a_j$ red, and the cycles in which $i_j > a_j$ blue. Call the obtained fixed point-free permutation with bicolored cycles $g(a)$.

It is then clear that g maps into the set D_n of fixed-point free permutations on $[2n]$ whose cycles are colored red or blue. The right-hand side of (3) counts precisely such involutions. Finally, it is straightforward to see that $g : A_n \rightarrow D_n$ is a bijection as it has an inverse. (Just choose the smaller entry in each of the red cycles and the larger entry in each of the blue cycles to recover i_1, i_2, \dots, i_n .) This completes the proof of (3), and therefore, of Proposition 1. \diamond

3 When k is odd

If we want to find a combinatorial proof of Theorem 1 along the line of the proof of Proposition 1, we encounter several difficulties. First, inserting a new fixed point into a partial permutation can increase the length of its longest increasing subsequence, taking it thereby out of the set that is being counted. More importantly, equality (3) no longer holds if we replace $n!$ by $u_k(n)$ on its left-hand side, and $x(h)$ by the number of fixed point-free involutions with no increasing subsequences longer than k on its right-hand

side. Indeed, for $k = 2$ and $n = 3$, the left-hand side would be $\binom{2n}{n} u_2(3) = 20 \cdot 5 = 100$, while the right-hand side would be $10 + 15 \cdot 3 + 15 \cdot 3 + 10 = 110$.

It is surprising that for the case of odd k , fixed points, and fixed point-free involutions, turn out to be relevant again. We point out that we will be considering involutions without long *decreasing* rather than increasing subsequences.

Note that $y_k(r)$ is equal to both the number of *involutions* on an r -element set with no *increasing* subsequences longer than k , and the number *involutions* on an r -element set with no *decreasing subsequences* longer than k (just take conjugates of the corresponding Standard Young Tableaux). However, this symmetry is broken if we restrict our attention to *fixed point-free involutions*, since the conjugate of a tableaux with even columns only may have odd columns, and our claim follows from Theorem 2.

Let $x_k(r)$ be the number of *fixed point-free* involutions of length r with no *decreasing subsequences* with more than k elements. Note that $x_k(r) = 0$ if r is odd.

Theorem 3 *For all positive integers n , and for all odd positive integers k the equality*

$$\sum_{r=0}^{2n} \binom{2n}{r} x_k(r) x_k(2n-r) = \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} y_k(r) y_k(2n-r) \quad (4)$$

holds.

Proof: Recall from the proof of Proposition 1 that B_n is the set of ordered pairs (p, q) , where p is an involution on a subset s_p of $[2n]$, and q is an involution on the set $[2n] - s_p$, the complement of s_p in $[2n]$.

Let $B(n, k, r)$ be the subset of B_n consisting of pairs (p, q) so that neither p nor q has a decreasing subsequence longer than k . Note that here p is an involution of length r and q is an involution of length $2n - r$. It is then clear that

$$|B(n, k, r)| = \binom{2n}{r} y_k(r) y_k(2n-r).$$

Let $B(n, k) = \cup_r B(n, k, r)$.

Recall the involution f from the proof of Proposition 1, (the involution that took the largest fixed point present in $p \cup q$ and moved it to the other involution), and let $f_{n,k}$ be the restriction of f to the set $B(n, k)$.

Our theorem will be proved if we can show that $f_{n,k}$ maps into $B(n, k)$. Indeed, that would show that the only pairs $(p, q) \in B(n, k)$ whose contribution to the right-hand side of (4) is not canceled by the contribution of

$f_{n,k}(p, q)$ are the pairs for which $f(p, q)$ is not defined. It follows from the definition of $f_{n,k}$ that these are the pairs in which both p and q are fixed point-free involutions.

Our main tool is the following lemma.

Lemma 1 *Let w be an involution whose longest decreasing subsequence is of length $2m + 1$. Then each longest decreasing subsequence of w must contain a fixed point.*

Proof: We use induction on z , the number of fixed points of w . If $z = 0$, then the statement is vacuously true, since by Theorem 2 the Standard Young Tableau corresponding to w has no odd columns, so the length of its first column (and so, the length of the longest decreasing subsequence of w) cannot be odd.

Otherwise, assume that we know that the statement holds for $z - 1$. Also assume that w has $z > 0$ fixed points, and w has a longest decreasing subsequence s of length $2m + 1$ that does not contain any fixed points. Remove a fixed point from w to get w' . Then w' still has a longest decreasing subsequence s of length $2m + 1$ that contains no fixed points, even though w' has only $z - 1$ fixed points, contradicting our induction hypothesis. \diamond

Let $(p, q) \in B(n, k)$. In order to show that $f_{n,k}$ maps into $B(n, k)$, we need to show that $f_{n,k}(p, q) = f(p, q) = (p', q')$ has no decreasing subsequence longer than k . The action of f on (p, q) consists of taking a fixed point of one of p and q and adding it to the other. We can assume without loss of generality that a fixed point of p is being moved to q . So the longest decreasing subsequence of p' is not longer than that of p , and so, not longer than k since p' is a substring of p . There remains to show that the longest decreasing subsequence of q' is also not longer than k .

As q' differs from q only by the insertion of the fixed point $M = M(p, q)$, the only way q' could possibly have a decreasing subsequence longer than k would be when q itself has a decreasing subsequence of length $k = 2m + 1$. In that case, by Lemma 1, all maximum-length decreasing subsequences of q contain a fixed point. So when M is inserted into q , and q' is formed, M cannot extend any of the maximum-length decreasing subsequences of q because that would mean that *two* fixed points are part of the same decreasing subsequence. That is impossible, since fixed points form increasing subsequences.

So indeed, $f_{n,k}$ maps into $B(n, k)$, and our claim is proved. \diamond

3.1 The special case $k = 3$

The first special case of Theorem 3 is when $k = 1$. Then $x_k(r) = 0$ for any r , while $y_k(r) = 1$ for any r . So (4) simplifies to the well-known binomial-coefficient identity

$$0 = \sum_{r=0}^{2n} (-1)^r \binom{2n}{r}.$$

The special case of $k = 3$ is more interesting. We point out that in this case, it is known [3] that $y_3(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} C_i$, where $C_i = \binom{2i}{i}/(i+1)$ is the i th Catalan number. The numbers $y_3(n)$ are called the *Motzkin numbers*.

It follows from Theorem 2 that if v is fixed-point free, then $RS(v)$ has no odd columns. Therefore, $x_{2m+1}(r) = x_{2m}(r)$. In particular, for $k = 3$, Theorem 3 simplifies to

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} y_3(r) y_3(2n-r) = \sum_{r=0}^{2n} \binom{2n}{r} x_2(r) x_2(2n-r).$$

Note that $x_2(r)$ is just the number of Standard Young Tableaux in which each column is of length two (of even length not more than two). The number of such tableaux is well-known (see for instance Exercise 6.19.wv of [8]) to be the Catalan number $C_{r/2}$ if r is even, and of course, 0 if r is odd. Therefore, the previous displayed equation simplifies to

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} y_3(r) y_3(2n-r) = \sum_i \binom{n}{i} \binom{2n}{2i} C_i C_{n-i}.$$

It turns out that the right-hand side is a well-known sequence. It is sequence A005568 in [7]. In particular, it is proved in [4], that the n th element f_n of this sequence has the closed form $f_n = C_n C_{n+1}$. Furthermore, it is shown in [3] that $f_n = y_4(2n)$.

So we have proved the following identity.

Corollary 1 *For all positive integers n , we have*

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} y_3(r) y_3(2n-r) = y_4(2n) = C_n C_{n+1} = \frac{\binom{2n}{n} \binom{2n+2}{n+1}}{(n+1)(n+2)}.$$

Acknowledgment

I am indebted to all three referees of this manuscript for their corrections and elucidating comments.

References

- [1] J. S. Beissinger, Similar constructions for Young tableaux and involutions, and their application to shiftable tableaux. *Discrete Math.* **67** (1987), no. 2, 149–163.
- [2] M. Coleman, R. Smith, *personal communication*, 2008.
- [3] D. Gouyou-Beauchamps, Standard Young Tableaux of height 4 and 5. *European J. Combin.*, **10** (1989), no. 1, 69–82.
- [4] R. K. Guy, Catwalks, Sandsteps and Pascal Pyramids, *J. Integer Seqs.*, **3** (2000), paper 00.1.6.
- [5] B. Sagan, The symmetric group. Representations, combinatorial algorithms, and symmetric functions. Second edition. Graduate Texts in Mathematics, **203**. Springer-Verlag, New York, 2001.
- [6] M.-P. Schützenberger, La correspondance de Robinson. Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976), 59–113. Lecture Notes in Math., Vol. 579, Springer, Berlin, 1977.
- [7] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, www.research.att.com/~njas/sequences.
- [8] R. Stanley, *Enumerative Combinatorics, Volume 2*, Cambridge University Press, Cambridge UK, 1999.
- [9] H. Wilf, Ascending subsequences of permutations and the shapes of tableaux. *J. Combin. Theory Ser. A*, **60** (1992), no. 1, 155–157.