A factorization theorem for m-level rook placements

by

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Let $B = (b_1, b_2, \ldots, b_n)$ be an integer partition where the parts are listed in weakly increasing order. We also consider B as a Ferrers board where b_j is the height of column j and the columns are bottom justified. Letting $r_k(B)$ denote the number of placements of k nonattacking rooks on B and $x\!\downarrow_k = (x)(x-1)\cdots(x-k+1)$, we have the famous Factorization Theorem of Goldman-Joichi-White which states that

$$\sum_{k\geq 0} r_k(B)x\!\downarrow_{n-k} = \prod_j (x + b_j - j + 1).$$

Briggs and Remmel considered a generalization of rook placements to m-level rook placements which are related to wreath products $C_m \wr S_N$ where C_m is a cyclic group and S_N a symmetric group. Ordinary rook placements correspond to the case $m = 1$. They were able to prove a version of the Factorization Theorem in this setting, but only for certain Ferrers boards. We give a generalization which holds for all Ferrers boards. Connections are also made with permutation statistics, q,t-Catalan numbers, and hyperplane arrangements. This is joint work with Kenneth Barrese, Nicholas Loehr and Jeffrey Remmel.