- We start we a data stream $x_0, x_1, \ldots, x_N$ of real numbers (sound file, ~~sto~~ Dow-Jones each minute, etc)

- We are interested in what frequencies or periodicities are present in the data stream

- If the data stream was given as a continuous function $f(t)$ we could find the Fourier Series as in the last lecture and use the coefficients as the amplitudes of each subharmonic

- But we have a discrete data set and so we need a discrete Fourier basis

- Treat the data as a vector (column)

$$\vec{x} = [x_0, x_1, \ldots, x_{N-1}]^T$$

(notice indexing - start at time zero)

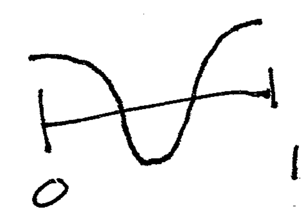
- For simplicity we restrict the sample times to the interval $[0,1]$ and assume we have sampled at uniform intervals and have N samples starting with 0

- So the sample times are $0, 1/N, 2/N, \ldots$ or at times $t_j = j/N$ for $j = 0, \ldots, N-1$

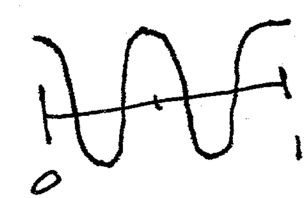
- On $[0,1]$ we transform the Fourier basis (ignoring normalizations) to

$$\varphi_k = e^{2\pi i k t} = \cos(2\pi k t) + i \sin(2\pi k t)$$

just looking at the real part $k=1$



$k=2$



· Now we sample the $\varphi_k(t)$ at the times t_j to get vectors

$$\varphi_k = \left[\varphi_k(t_0), \varphi_k(t_1), \ldots, \varphi_k(t_{N-1})\right]^T$$

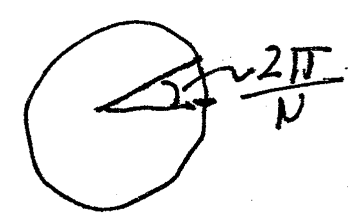
$$= \left[e^{2\pi i k t_0}, e^{2\pi i k t_1}, \ldots, e^{2\pi i k t_{N-1}}\right]^T$$

$$= \left[e^{2\pi i k 0/N}, e^{2\pi i k 1/N}, \ldots, e^{2\pi i k\, N-1/N}\right]^T$$

$$= \left[\left(e^{\frac{2\pi i}{N}}\right)^{k\cdot 0}, \left(e^{\frac{2\pi i}{N}}\right)^{k\cdot 1}, \ldots, \left(e^{\frac{2\pi i}{N}}\right)^{k(N-1)}\right]^T$$

$$= \left[(\omega_N)^{k\cdot 0}, (\omega_N)^{k\cdot 1}, \ldots, (\omega_N)^{k(N-1)}\right]^T$$

where $\omega_N = e^{2\pi i/N}$, the N^{th} root of unity



More succinctly, the j^{th} component of $\vec{\varphi}_k$ is

$$(\vec{\varphi}_k)_j = \omega_N^{kj}$$

It turns out that the $\{\vec{\varphi}_k\}$ are orthogonal but not orthonormal so we define for $k = 0, .., N-1$

$$\vec{z}_k = \frac{1}{\sqrt{N}}\left[1, \omega_N^{k}, \omega_N^{2k}, .., \omega_N^{(N-1)k}\right]^T$$

__Theorem__ (HW): The $\{\vec{z}_k\}$ form an orthonormal basis for $\mathbb{C}^N$

__Examples__: For any N,

$$\vec{z}_0 = \frac{1}{\sqrt{N}}[1, 1, .., 1]^T$$

$\boxed{N=4}$ $\quad \omega = w_4 = e^{2\pi i/4} = e^{\pi i/2} = i$

$$\vec{q}_3 = \frac{1}{2}[1, \omega^3, \omega^6, \omega^9]^T$$

$$= \frac{1}{2}[1, \omega^3, \omega^2, \omega^1]^T$$

basic fact if $k = 4j + l \Rightarrow \omega_4^k = \omega^{4j+l}$

$= (\omega^{4j})\,\omega^l = \omega^l$. so $\omega^6 = \omega^{4+2} = \omega^2$

$$\text{So } \vec{q}_3 = \frac{1}{2}[1, -i, -1, i]^T$$

$\boxed{N=6}$ $\quad w_6 = \omega = e^{2\pi i/6} = e^{\pi i/3}$

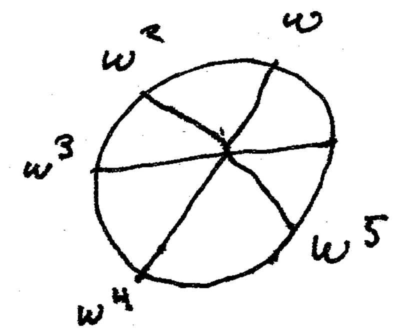
$$\vec{q}_5 = \frac{1}{\sqrt{6}}[1, \omega^5, \omega^{10}, \omega^{15}, \omega^{20}, \omega^{25}]^T$$

$$= \frac{1}{\sqrt{6}}[1, \omega^5, \omega^4, \omega^3, \omega^2, \omega]$$

$$= \frac{1}{\sqrt{6}}[1,$$

$$= \frac{1}{\sqrt{6}}\left[1, e^{5\pi i/3}, e^{4\pi i/3}, e^{3\pi i/3}, e^{2\pi i/3}, e^{\pi i/3}\right]^T$$

$$= \frac{1}{\sqrt{6}}\left[1, \frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i\right]^T$$



How do we change our data vector $\vec{x}$ into the ~~Fourier~~ Discrete Fourier Basis $\{\vec{q}_k\}$?

Recall our basis formula when $\{\vec{q}_0, \ldots, q_{N-1}\}$ is an orthonormal basis

If $\vec{v} = \alpha_0 \vec{q}_0 + \dots + \alpha_{N-1} \vec{q}_{N-1}$

then $\alpha_k = \langle \vec{q}_k, \vec{v} \rangle = \bar{\vec{q}}_k^{\,T} \vec{v}$

Now making the coordinates or amplitudes α_k into a vector $\vec{\alpha} = [\alpha_0, \alpha_1, \dots, \alpha_{N-1}]^T$

then

$$\underbrace{\begin{bmatrix} \bar{\vec{q}}_1^{\,T} \\ \bar{\vec{q}}_2^{\,T} \\ \vdots \\ \bar{\vec{q}}_{N-1}^{\,T} \end{bmatrix}}_{A} \vec{v} = \vec{\alpha}$$

This matrix is the Fourier matrix F

The K^{th} row of F is

$$\vec{z}_K^{\,T} = \frac{1}{\sqrt{N}}\left[1, \bar{\omega}_N^{K}, \bar{\omega}_N^{2K}, \ldots, \bar{\omega}_N^{(N-1)K}\right]$$

$$= \frac{1}{\sqrt{N}}\left[1, \omega_N^{-K}, \omega_N^{-2K}, \ldots, \omega_N^{-(N-1)K}\right]$$

So we see the entries of F are

$$F_{ij} = \frac{\omega^{-ij}}{\sqrt{N}} \quad \text{where } i = 0, \ldots, N-1, \; j = 0, \ldots, N-1$$

Notes $F_{ji} = F_{ij}$ so $F^T = F$, Symmetrix

Also lets look at $Q = F^*$, the adjoint

$Q = F^* = \begin{bmatrix} \vec{q}_0 \cdots \vec{q}_{N-1} \end{bmatrix}$ has its columns

an orthonormal basis, so F^* is unitary

or Q is unitary or $Q^* = Q^{-1}$

so $(F^*)^* = (F^*)^{-1}$ or $F = (F^*)^{-1}$ or

$F^{-1} = F^*$ so F is unitary also.

Example: $N = 4$, $\omega_4 = e^{2\pi i/4} = e^{\frac{2\pi i}{2}} = i$

$$Q = \frac{1}{2}\begin{bmatrix} w^{0\cdot 0} & w^{0\cdot 1} & w^{0\cdot 2} & w^{0\cdot 3} \\ w^{1\cdot 0} & w^{1\cdot 1} & w^{1\cdot 2} & w^{1\cdot 3} \\ w^{2\cdot 0} & w^{2\cdot 1} & w^{2\cdot 2} & w^{2\cdot 3} \\ w^{3\cdot 0} & w^{3\cdot 1} & w^{3\cdot 2} & w^{3\cdot 3} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

So $F = Q^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$

- It is also useful to express this in coordinates
- We now switch to more standard notation in the field: If x is the data (notice no arrow) then its Discrete Fourier transform is

$$\hat{x} = F x = DFT(x)$$

In coordinates

$$\hat{x}_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k \, \omega_N^{-kj}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k \, e^{-2\pi i kj/N}$$

for $j = 0, \dots, N-1$.

Also, sometimes $DFT(x) = \underline{X}$, capital letters.

What about going back from the DFT to x?

(This is called going from frequency domain to the time domain)

$$\begin{tikzcd} \boxed{\text{Time Domain}} \arrow[r, bend left, "\text{DFT}"] & \boxed{\text{Frequency Domain}} \arrow[l, bend left, "\text{IDFT}"] \end{tikzcd}$$

Since $\hat{x} = Fx$, $x = F^{-1}\hat{x} = F^{*}\hat{x} = Q\hat{x}$

F is unitary

So for $j = 0, \ldots, N-1$

$$x_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{x}_k \, \omega_N^{kj}, \quad x = \text{IDFT}(\hat{x})$$

notice similarity to forward transform.

There are many other considerations and conventions

(1) In practise, $\hat{x} = Fx$ is not computed by matrix multiplication, but rather by the very clever, much faster Fast Fourier Transform

(2) While the $1/\sqrt{N}$ is nice in both DFT and IDFT for symmetry and the Linear Algebra (orthonormal basis) in practise a single $1/N$ is put on the DFT or on the IDFT

(3) Matlab uses index $1, \ldots, N$ and puts the $\frac{1}{N}$ on the IDFT

so if $X = \mathrm{fft}(x)$

$$X_j = \sum_{k=1}^{N} x_k \, \omega^{-(k-1)(j-1)} \qquad \omega = e^{2\pi i/N}$$

$$x_j = \frac{1}{N} \sum_{k=1}^{N} X_k \, \omega^{(k-1)(j-1)}$$