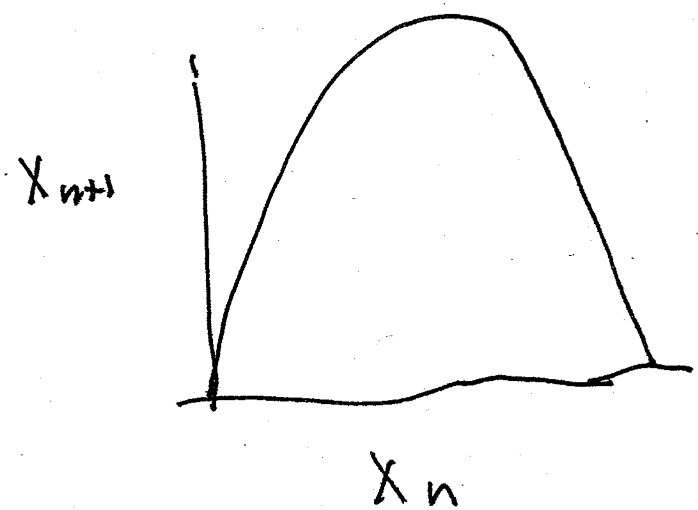
Informal overview to start

A simple population model as a Dynamical System

X_n = population at time n (years)

$X_{n+1} = f(X_n)$ Rule that governs population change year to year



Logic: if a few $\Rightarrow$ increase
if too many $\Rightarrow$ finite resources force decrease

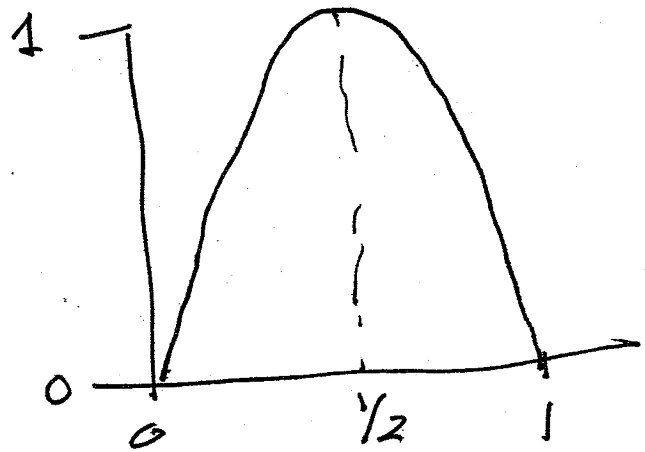
- $x_1 = f(x_0)$, $x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$

  $\ldots$ $x_n = \underbrace{f(f(f \ldots f}_{n\text{-times}}(x_0)) = f^n(x_0)$

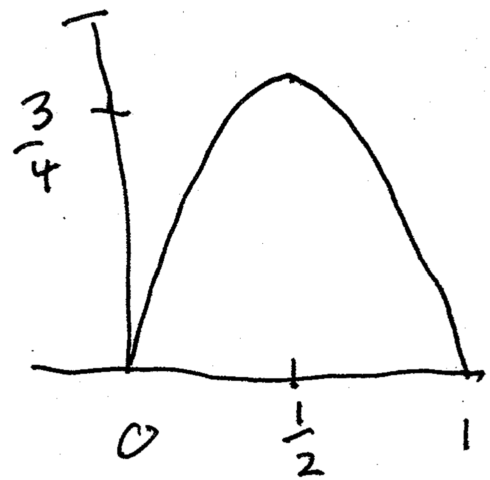
- Trajectory or orbit of x, $O(x,f) = \{x, f(x), f^2(x), \ldots\}$
  collects together entire future of x

- To be more concrete we rescale the range = domain to $I = [0,1]$ and use the formula $f(x) = \mu x(1-x)$

  $0 < \mu \leq 4$ So $f: I \to I$, self-map.



$\mu = 4$

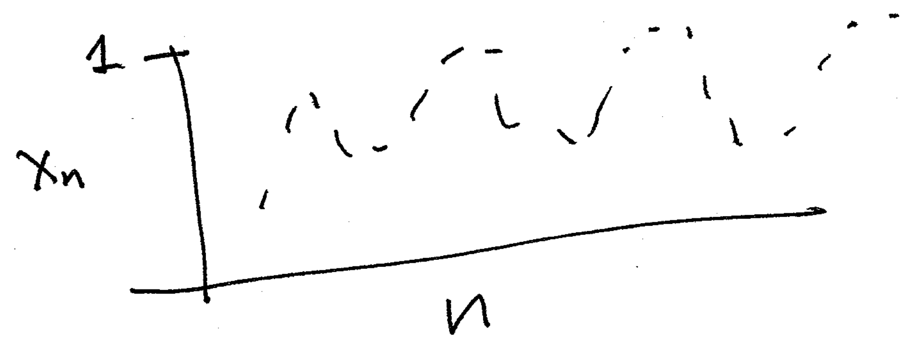


$\mu = 3$

single max at
$x = 1/2$
$[f'(x) = \mu(1-2x)]$

$\boxed{\text{MATLAB DEMO}}$

The graphs are produced by iterating random initial conditions for $N = 500$ iterates and plotting (n, x_n)



The argument is the parameter η in

$$f_\eta(x) = \eta\, x(1-x)$$

Shows importance of asymptotic or long term behavior. One way to formalize this is the omega-limit set

$$\omega(x) = \{ y : \exists\, n_i \to \infty \text{ with } f^{n_i}(x) \to y \}$$

eg 

period two point is ω-limit

ω limit sets can be fixed points 

periodic points 

or more complicated sets.

or the whole space.

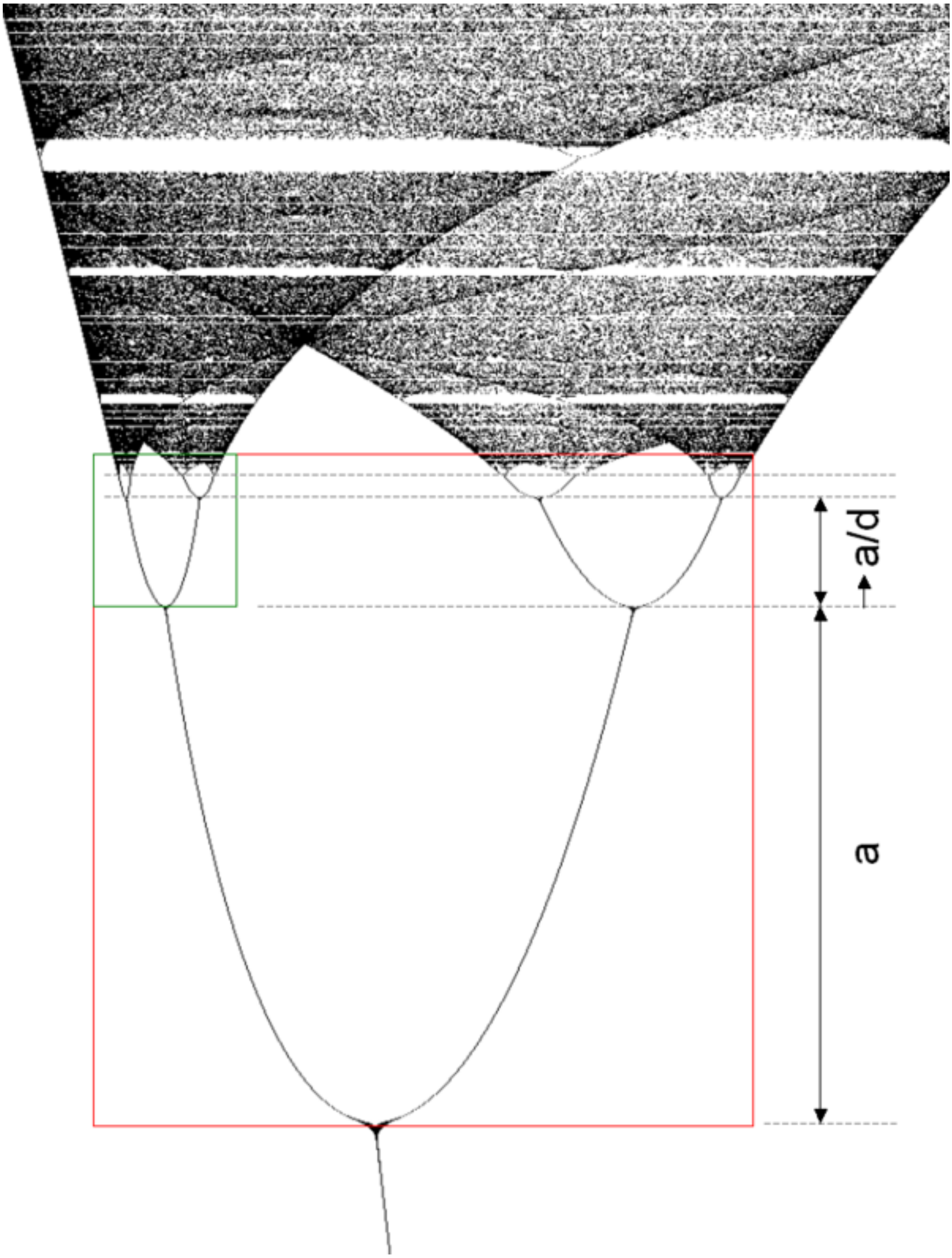
What is the "typical" ω-limit set

measure, topology

Next Picture Shows the bifurcation diagram as μ is varied

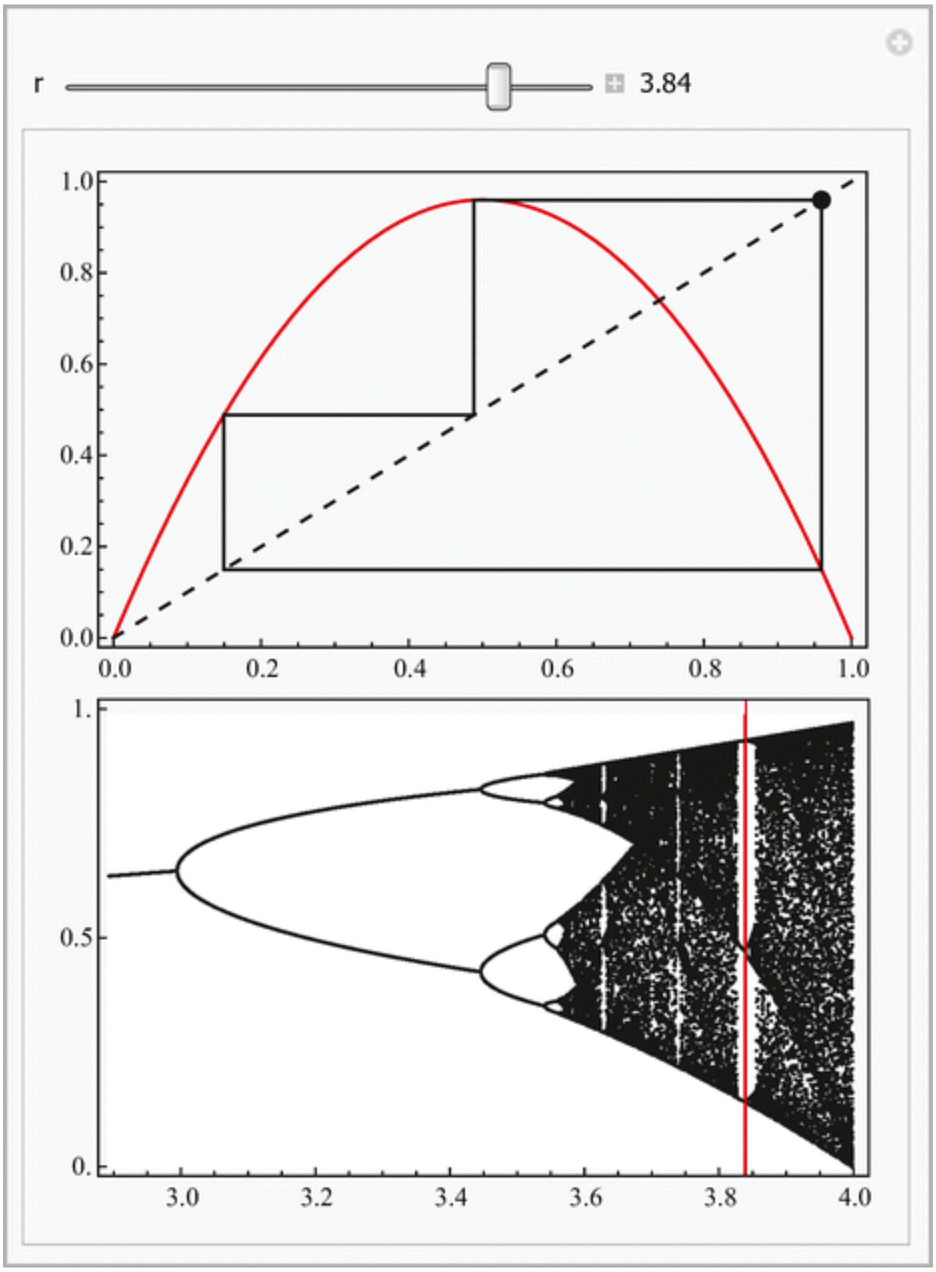
For each vertical slice or μ value a random point is chosen and the first few iterates (the transients) are not shown

This gives an approximate picture of $\omega(x)$ for the random initial point x



Wikipedia

The next picture is a "period 3 window" in the bifurcation diagram



Wagon, Mathematics in Action

MORE FORMAL DEFINITION

- Nicer to work with $h: X \to X$ with X compact metric and h a homeomorphism (bijective, bi-continuous)

  orbits are then $o(x,h) = \{\ldots, h^{-2}(x), h^{-1}(x), x, h(x), h^2(x), \ldots\}$

  with $h^{-n} = (h^{-1})^n = h^{-1} \circ h^{-1} \circ \ldots \circ h^{-1}$

- This is most nicely described as a group action of $\mathbb{Z}$ on X or

  $H: \mathbb{Z} \times X \to X$ is continuous and

  (1) $H(0,x) = x$

  (2) $H(n+m, x) = H(n, \cancel{H_m(x)})$ H(m,x)

Sometimes the action is written as

$$n \cdot x = H(n, x)$$

Satisfies (1) $0 \cdot x = x$

(2) $(n+m) \cdot x = n \cdot (m \cdot x)$

Given a $\mathbb{Z}$-action the generator is $H(1, x) = h(x)$ then (exercise)

$$H(n, x) = h^n(x).$$

Even fancier, H defines a homomorphism

$$\mathbb{Z} \to \mathrm{Homeo}(X)$$

with the latter a group under composition

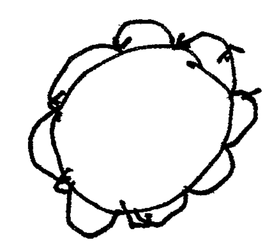
~~Exam~~ We will usually be simple and just designate the invertible, discrete dynamical system as a pair (X, h).

Example $S^1 = \mathbb{R}/\mathbb{Z}$ (integers mod 1 = circle)

$R_\omega : S^1 \circlearrowleft$ is $R_\omega(\theta) = \theta + \omega \mod 1$

FACTS:

$\omega = p/q \in \mathbb{Q}$ every orbit is periodic



$\omega \notin \mathbb{Q}$, irrational, every orbit is dense in S^1

Our initial example $f_\mu : I \circlearrowleft$ is not invertible and we can only iterate forward so it yields a semi-group action

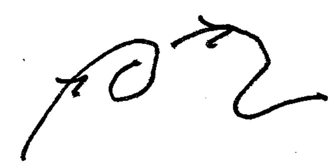
$$\mathbb{N} \times I \to I$$

The final important example is a flow or $\mathbb{R}$-action

$$\varphi : \mathbb{R} \times X \to X$$

$$\varphi(0,x) = x$$
$$\varphi(t+s,x) = \varphi(t,\varphi(s,x))$$

These come from soln to differential equations when φ is differentiable $\left.\frac{\partial \varphi(x,t)}{\partial t}\right|_{t=0} = \vec{F}(x)$, the vector field

orbits are curves $o(x,\varphi) = \{\varphi(t,x) : t \in \mathbb{R}\}$ 

$\varphi(t,x)$ usually written $\varphi_t(x)$ [more later] (LORENZ)