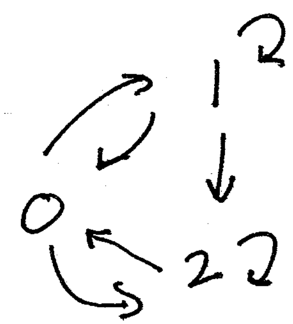
D810

1

SSFT contiued

- A is $n \times n$ $\{0,1\}$ matrix
- Def: A is a transition matrix if it has no row of all zeros and no column of zeros.
  This means that every symbol has a successor and a predecessor.
- $\Sigma_A^+ = \{ \underline{s} \in \Sigma_n^+ : A_{s_i s_{i+1}} = 1, \forall i \}$
- $A \longleftrightarrow$ directed graph on n vertices, $i \to j \iff A_{ij} = 1$
- 

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

- $b = s_0 s_1 \ldots s_{n-1}$ is an allowable word of length n if every $s_i s_{i+1}$ is allowable

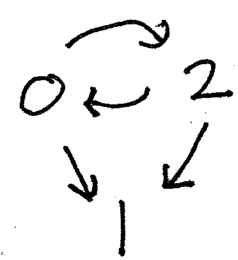
- $\mathcal{B}$ allowable words $\longleftrightarrow$ finite paths in graph

- $\underline{s} \in \Sigma_A^+ \longleftrightarrow$ infinite paths in graph

- A transition matrix says $\forall i \; \exists j$ with $i \to j$
  $\forall j \; \exists i \quad i \to j$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

not transition matrix



$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

not transition matrix

no pure "sources or sinks"

- **Question:** How do properties of A or its graph correspond to Dynamical properties of (Σ_A, σ)

- When is (Σ_A, σ) transitive?

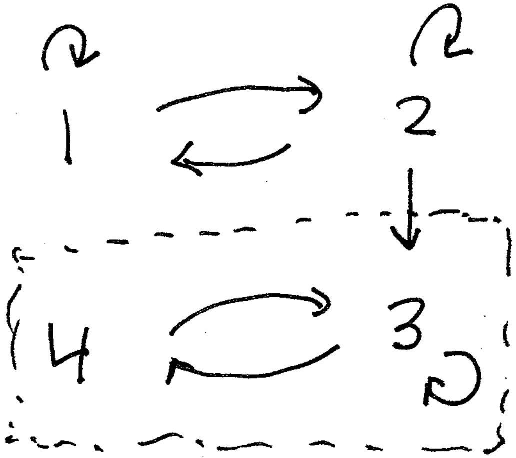
- In the graph the intuition is transitive $\Leftrightarrow$ you can get from anywhere to anywhere else.

- DEF: A is a $n \times n$ $\{0,1\}$-matrix

A is reducible if $\exists\, i,j$ so that for all $n > 0$ $(A^n)_{ij} = 0$. In the graph there is no path of any length

$\cdot\, j$

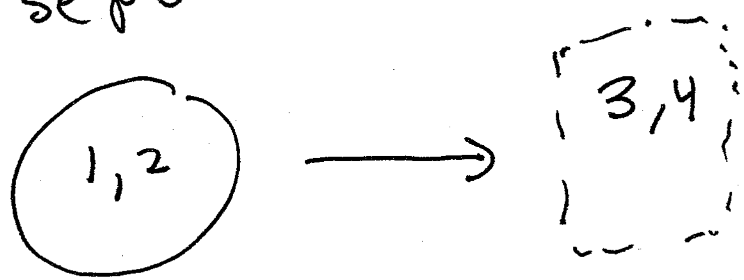
$\cdot\, i$



$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

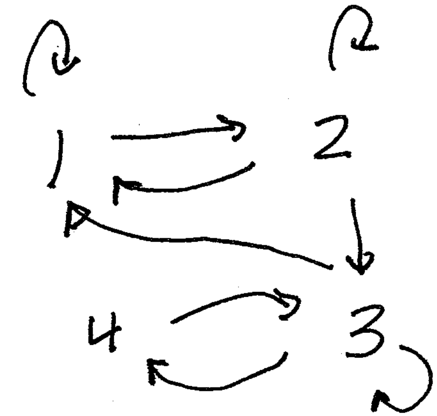
- It is harder to check reducibility in the matrix, but in the graph There is no way to get from (3 or 4) to 2 or 1 [I changed labeling to 1, 2, 3, 4]

- The name reducible comes since I can seperate out the dotted sussystem



and reduce to 2 systems. with one "absording"

- A is irreducible if $\forall i,j\ \exists n$ so that $(A^n)_{ij} \neq 0$, so for any two pair of vertices there is a path from one to the other of some length.



$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is irreducible

- If $\exists N$ $A^N > 0 \Rightarrow$ called primitive $\Rightarrow$ irreducible

- We need one more fact: If b is a block $b = s_0 s_1 \dots s_{n-1}$, the <u>cylinder set</u> for the block b is

$$C_b = \{ \underline{t} \in \Sigma_n^+ : t_0 = s_0, \dots, t_{n-1} = s_{n-1} \}$$

- FACT: C_b is open and closed.
- <u>Theorem</u>: Assume A is a transition matrix (Σ_A^+, σ) is transitive (has a dense orbit) if and only if A is irreducible.

Proof For a block $b = b_0 \ldots b_{n-1}$ let

$B(b) = b_0$ (beginning) $E(b) = b_{n-1}$ (end)

Assume A is irreducible. This implies that for all i, j $\exists n$ with $(A^n)_{ij} \neq 0$. Thus there is a path from i to j of length n and thus a block c_{ij} with $B(c) = i$ and $E(c) = j$.

Let b_1, b_2, b_3 be an enumeration of the allowable blocks of Σ_A^+. And let

$$s = b_1 t_1 b_2 t_2 b_3 t_3 \ldots$$

~~where $B(t_1) = E(b_1)$~~

where t_k is constructed from C_{ij} by stripping of the first and last symbols where $i = E(b_k)$ and $j = B(b_{k+1})$

<u>Example</u>: Say $b_7 = 012$ $b_8 = 597$

$C_{25} = 26435 \Rightarrow t_7 = 643$ and

$b_7 t_7 b_8 = 01264359 7$ will be allowable. Note it could happen that t_k is the empty word if C_{ij} has length 1 or 2

Now remember $\underline{s}, \underline{t}$ are close in Σ_n^+
if they agree for a long way.
Given $\underline{w}$, $\varepsilon > 0$ let N be such that $u_i = v_i$ for $i = 0, .., N$
implies $d(\underline{u}, \underline{v}) < \varepsilon$. Now since we have
enumerated all the allowable blocks, $\exists\, b_M$ so that
$(b_M)_i = w_i$ for $i = 0, ., N$ and some ℓ
$\sigma^{\ell}(\cancel{\theta}\,\underline{s}) = b_M \,\vdots\, \ldots\ldots$. Thus
so that
$d(\sigma^{\ell}(\underline{s}), \underline{w}) < \varepsilon$ and so $o(\underline{s}, \sigma)$ is dense
in Σ_A^+.

Now conversely, assume that (Σ_A^+, σ) has a dense orbit say $o^+(\underline{s}, \sigma)$. Now we claim that $\forall\, k>0$, $o^+(\sigma^k(\underline{s}), \sigma)$ is dense. You finish he proof using he claim and then prove fe claim. Given i, j consida he cylinder sets C_i and C_j. These are open, so $\exists\, n_1$ with $\sigma^{n_1}(\underline{s}) \in C_i$. Now by he claim $o(\sigma^{n_1+1}(\underline{s}), \sigma)$ is also dense. Thus $\exists\, n_2 > n_1$ with $\sigma^{n_2}(\underline{s}) \in C_j$. Let b be he first $n_2 - n_1 + 1$ entries of $\sigma^{n_1}(\underline{s})$ $= i, *, *, \ldots, *, j$ which yields an allowable path $i \to j$ so $(A^{n_2-n_1+1})_{ij} \neq 0$ [we used he fact that $\underline{t} \in C_i \Rightarrow \underline{t} = i***\ldots$]

Now we prove the claim. Assume $\overline{O(\sigma^k(\underline{s}, \sigma))}$ is not dense. First note that if $\sigma^{n_i}(\underline{s}) \to \underline{t}$ as $n_i \to \infty$ then $\sigma^{n_i}(\underline{s}) \to \underline{t}$ for $n_i \geq k$. This implies that the only points not in $\overline{O(\sigma^k(\underline{s}), \sigma)}$ are the points $\underline{s}, \sigma(\underline{s}), \ldots, \sigma^{k-1}(\underline{s})$. As in the proof in class, these points must be isolated (if not $\exists\, \underline{t}^{(n)} \to \underline{s}$ say with $\underline{t}^{(n)} \in \overline{O(\sigma^k(\underline{s}), \sigma)}$ and so $\underline{s} \in \overline{O(\sigma^k(\underline{s}), \sigma)}$).

Since $\underline{s}$ is isolated, $\exists\, \varepsilon > 0$ so that the only point in $B_\varepsilon(\underline{s}) \cap \Sigma_A^+$ is $\underline{s}$ itself. This implies that $\exists\, N$ so that $\underline{t} \in \Sigma_A^+$ and $s(0:N) = t(0:N)$ $\Rightarrow \underline{t} = \underline{s}$. This means that for every s_i with $i > N$, s_i has exactly one allowable successor. Thus $\underline{s}$ is eventually periodic, and since it is dense, it must be periodic.

The claim holds trivially of $\Sigma_A^+ = \overline{O(\underline{s}, \sigma)}$ is a periodic orbit.