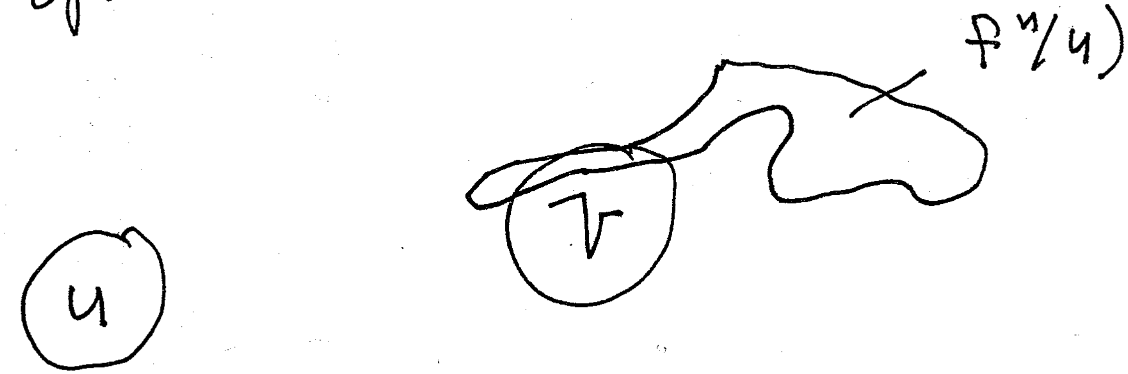
A proof similar to that at the end of last time yields

<u>Theorem:</u> If A is irreducible then either Σ_A^+ is a single periodic orbit or Σ_A^+ is perfect (every point is a limit point, i.e. no isolated points).

$f: X \to X$ is called <u>topologically mixing</u> if for all open U, V $\exists N$ so that $f^n(U) \cap V \neq \emptyset$ $\forall n \geq N$

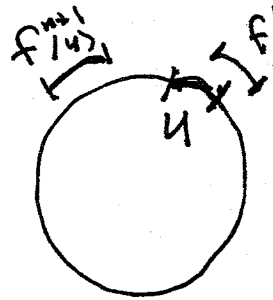


Transitivity requires $\exists n$ with $f^n(U) \cap V \neq \emptyset$

So Top mixing $\Rightarrow$ transitivity. But not the converse.

$R_\alpha : S^1 \circlearrowleft$ $\quad R_\alpha(\theta) = \theta + \alpha \quad \alpha \notin \mathbb{Q}$ is minimal

So it is certainly transitive but let $U = V$, $|U| < \frac{\alpha}{4}$



then $f^n(U) \cap V \neq \emptyset$

$\Rightarrow f^{n+1}(U) \cap V = \emptyset$

f = R_\alpha

So it is not mixing.

[There is also something called mixing in ergodic theory, but for here mixing = top mixing

• Trivial example: Single periodic orbit is transitive not mixing.

- Recall a transition matrix A is called primitive if $\exists N$ so that $\forall i,j$ $(A^N)_{ij} \neq 0$. Note that since A has no zero rows or columns by definition, we have $\forall i,j$ $(A^n)_{ij} \neq 0$, $\forall n \geq N_0$

- Theorem: Assume A is a transition matrix. A is primitive $\Leftrightarrow$ Σ_A^+ is top mixing.

- For the proof we need two preliminary facts.

(1) Recall that $\mathcal{B} = \{U_\lambda\}_{\lambda \in \Lambda}$, each U_λ open is a base for a topology if every open set is the union of some U_λ.

- Eg: In a metric space $\{B_{\frac{1}{n}}(x) : x \in X, n = 1, 2, \ldots\}$ is a base.
- It is easy to see that if X has a base $\{U_\lambda\}_{\lambda \in \Lambda} = \mathcal{B}$ then (X, f) is top mixing $\iff$ for every pair of $U_1, U_2 \in \mathcal{B}$ $\exists N$ so that $f^n(U_1) \cap U_2 \neq \emptyset$, $\forall n \geq N$.
- For $b = (s_0, \ldots, s_{n-1})$ an allowable block for Σ_A^+ recall its cylinder set is

$$[b] = C_b = \{ t \in \Sigma_A^+ : t_i = s_i \text{ for } i = 0, \ldots, n-1 \}$$

alternative notation

- $\mathcal{B} = \{ C_b : b \text{ is allowable for } \Sigma_A^+ \}$ is a base for the topology of Σ_A^+

Example

$$\begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[loop above] \end{tikzcd}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} > 0$$

So $(\overrightarrow{\Sigma}_A, \sigma)$ is top mixing.

Proof assume $A^n > 0$ $\forall\ n \geq N$ (short hand notation)

Claim: for all allowable blocks b and c, and $n \geq N$ $\exists\ \underline{s} \in \Sigma_A^+$ with $\underline{s} = b \ldots\ldots c \ldots\ldots$ where $\sigma^n(\underline{s}) = c \ldots$

Given the claim we have $\underline{s} \in [b]$ and $\sigma^n(\underline{s}) \in [c]$ and we are done.

To prove the claim let i = last symbol is b and j the first symbol in c. Since $(A^n)_{ij} \neq 0$ when $n \geq N$ we may find an allowable block $d = i \ldots j$. Now cut off the beginning and end symbols of d to get d' and so $bd'c \ldots$ is allowable, proving the claim.

Now conversely, assume (Σ_A^+, σ) is top mixing

For each symbols i, j we may find an N_{ij}

so that $\sigma^n([i]) \cap [j] \neq \phi \ \forall\, n \geq N_{ij}$. Let

$N = \max_{i,j} \{N_{ij}\}$ so $\sigma^n([i]) \cap [j] \neq 0 \ \forall\, n \geq N$

and all i, j. Thus for all $i j$ there is a allowable path of

length n with $b = i \ldots j$ and so $(A^n)_{ij} \neq \phi$. $\blacksquare$

$\boxed{\text{Li + Yorke}}$ If $f: \mathbb{R} \to \mathbb{R}$ is continuous and has a period 3

point then it has periodic points of all periods.

(Period 3 implies Chaos)

L7

We need some preliminaries $f: \mathbb{R} \to \mathbb{R}$ is continuous

<u>Lemma</u> (a) If I is a closed interval and

$f(I) \supset I \Rightarrow \exists$ a point $p \in I$ with $f(p) = p$

(b) If $I_1, I_2, \ldots, I_n$ is a collection of intervals

with $f(I_j) \supseteq I_{j+1}$ $j = 1, \ldots, n-1$ and $f(I_n) \supseteq I_1 \Rightarrow \exists$

a point $p \in I_1$ with $f^n(p) = p$ and $f^i(p) \in I_{i+1}$

for $i = 0, \ldots, n-1$.

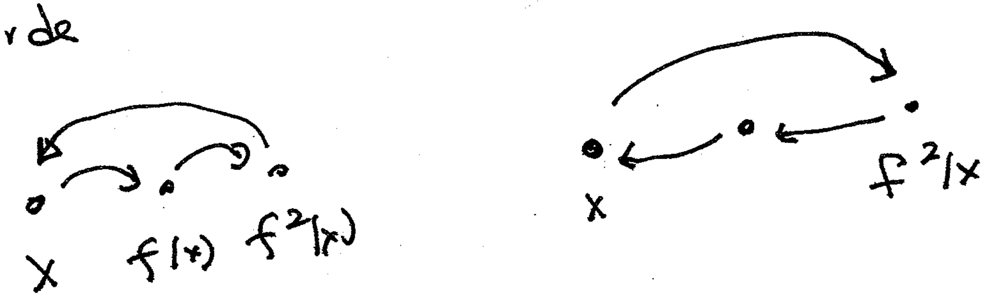
<u>Sketch proof</u> (a) I

Intermediate value Theorem

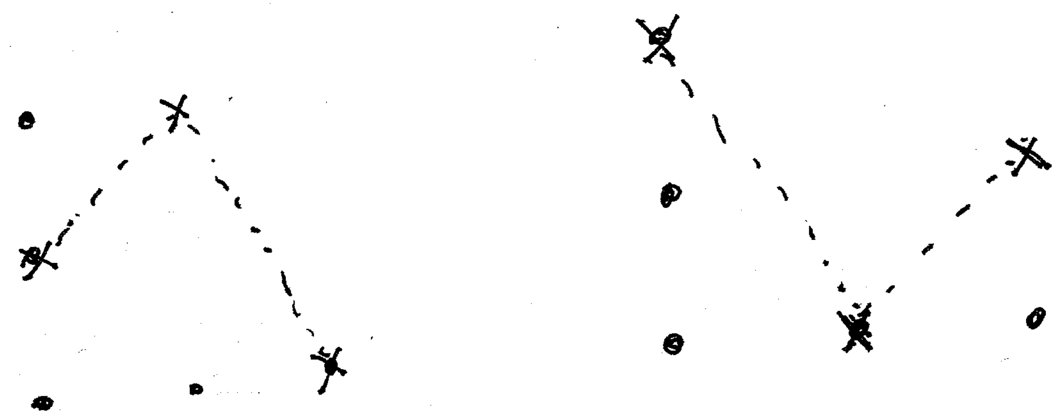
(b) Induction

Proof Assum x is periodic and reindexing $x < f(x)$, $x < f^2(x)$

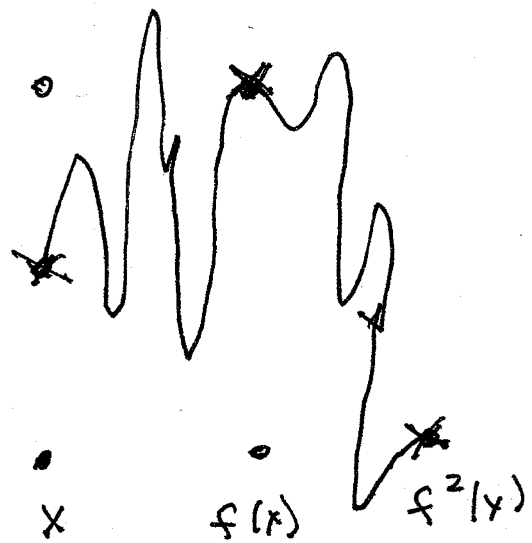
Then there are two possibilities in terms of permutations and orde



In a graphical terms



Now the actual function could do complicated things between points of the orbit but the crucial thing is that the dotted lines above are the simplest

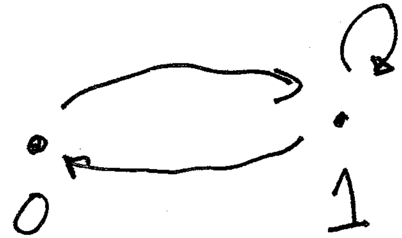


consider the left case,
the right case is similar

letting $I_0 = [x, f(x)]$ $I_1 = [f(x), f^2(x)]$

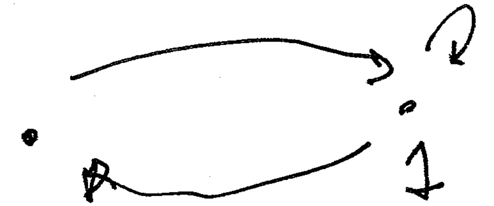
we have $f(I_0) \supseteq I_1$ $f(I_1) \supseteq I_0 \cup I_1$

or



and an allowable path $s_0 \dots s_{n-1}$ in this graph yields

$f(I_{s_0}) \supseteq I_{s_1}, \dots f(I_{s_{n-2}}) \supseteq I_{s_{n-1}}$.



Thus, in particular using our preliminarly lemma.

If $b = s_0 s_1 \ldots s_{n-1} s_0$ is an allowable path there is a periodic orbit $p \in I_{s_0}$ with $f^n(p) = p \in I_{s_0}$ and $f^i(p) \in I_{s_i}$ $i = 1, \ldots, n-1$. In other words, p has itinerary $\cdot (s_0 s_1 \ldots s_{n-1})^\infty$

Now for the proof. Given an n, let n \not= 3 $b = 0111\ldots10$ yielding p with itinerary $(0111\ldots1)^\infty$. Now $p \in Fix(f^n)$ and p cannot be on the endpoints of I_0 or I_1 since itinerary is wrong. Thus since b is not the repitition of a smaller length block, p has period n as required.

Remarks (1) There are many more period n points There are formulas using the trace but they get complicated.

(2) Sharkovski's order on $\mathbb{N}$ is

$3 \rhd 5 \rhd 7 \ldots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots \rhd 2^n 3 \rhd 2^n 5 \rhd 2^n 7 \rhd \ldots$

$\cdots \rhd 2^{n+1} \rhd 2^n \rhd \cdots \rhd 2^2 \rhd 2 \rhd 1$

<u>Theorem</u>: If $n \rhd p$ and $f: \mathbb{R} \to \mathbb{R}$ continuous has a period point of period $n \Rightarrow$ it has one with period p.

Proof is similar to period 3 but with more complicated combinatorics.