# Two sided shifts

- $\Sigma_n = \{ \underline{s} : \underline{s} = \dots s_{-1} s_0 s_1 s_2 \dots \} = \{0, \dots, n-1\}^{\mathbb{Z}}$
- $\sigma : \Sigma_n \to \Sigma_n$ $\quad \sigma(\underline{s}) = \dots s_{-2} s_{-1} s_0 . s_1 s_2 \dots$

  is now a homeomorphism.

- $\underline{s}, \underline{t}$ are now close if $s_i = t_i$ for $|i| < N$ large $N$

  so middles are the same

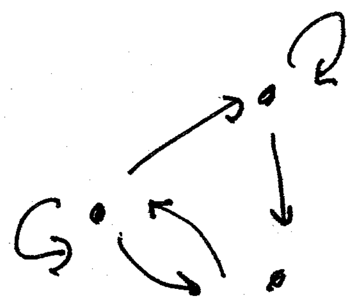
- $d(\underline{s}, \underline{t}) = \sum_{k \in \mathbb{Z}} \frac{|s_k - t_k|}{n^{|k|}}$ metric in $\Sigma_n$
- Now cyllnder sets specify where they start

  $[b]_k = \{ \underline{s} \in \Sigma_n : s_k = b_0 \dots s_{k+\ell} = b_\ell \}$

  where $b = b_0 \dots b_\ell$

- $A$ is transition matrix (no zero rows or zero columns)
- SSFT (two sided) $A$ is $(n \times n)$ transition matrix

$$\Sigma_A = \{ \underline{s} \in \Sigma_n : A_{s_i s_{i+1}} = 1 \ \forall_i \in \mathbb{Z} \}$$



Now $\Sigma_A \Leftrightarrow$ all bi-infinite allowable paths

- $\Sigma_A$ is compact and completely invariant $\sigma(\Sigma_A) = \Sigma_A$
- **Theorem:** $A$ is a transition matrix
  - $(\Sigma_A, \sigma)$ is transitive $\Leftrightarrow$ $A$ is irreducible
  - $(\Sigma_A, \sigma)$ is top. mixing $\Leftrightarrow$ $A$ is primitive
  - If $A$ is irreducible $\Rightarrow$ either $\Sigma_A$ is a single periodic orbit or it is a perfect set

- Smale's geometric Horseshoe is an example that has been central to modern dynamics:

- $f : S^2 \to S^2$ (2-sphere) whose recurrent set (closured recurrent points) consists of 3 pieces

  - $P_-$ a repelling fixed pt
  - $\Lambda$ a compact, "hyperbolic" set conjugate to $(\Sigma_2, \sigma)$ Cantor Set
  - $P_+$, an attracting fixed pts

for $x \notin \Lambda$ $\alpha(x) = P_-$, $\omega(x) = P_+$

and $\Lambda$ is $f$ invariant and

$f|\Lambda$ is trasitive, and so dynamicall irreducible

$\bullet\ P_-$

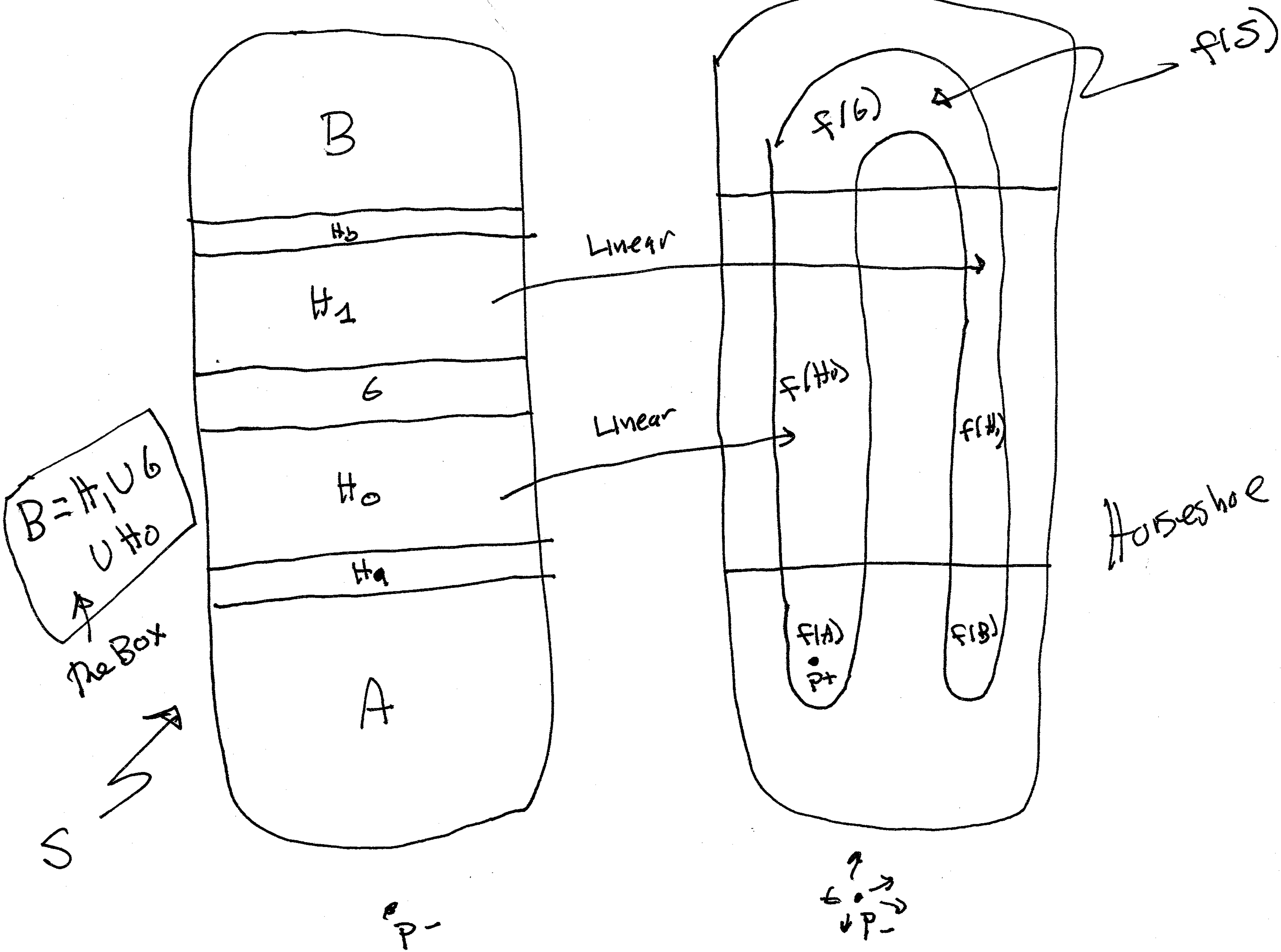
$\downarrow$

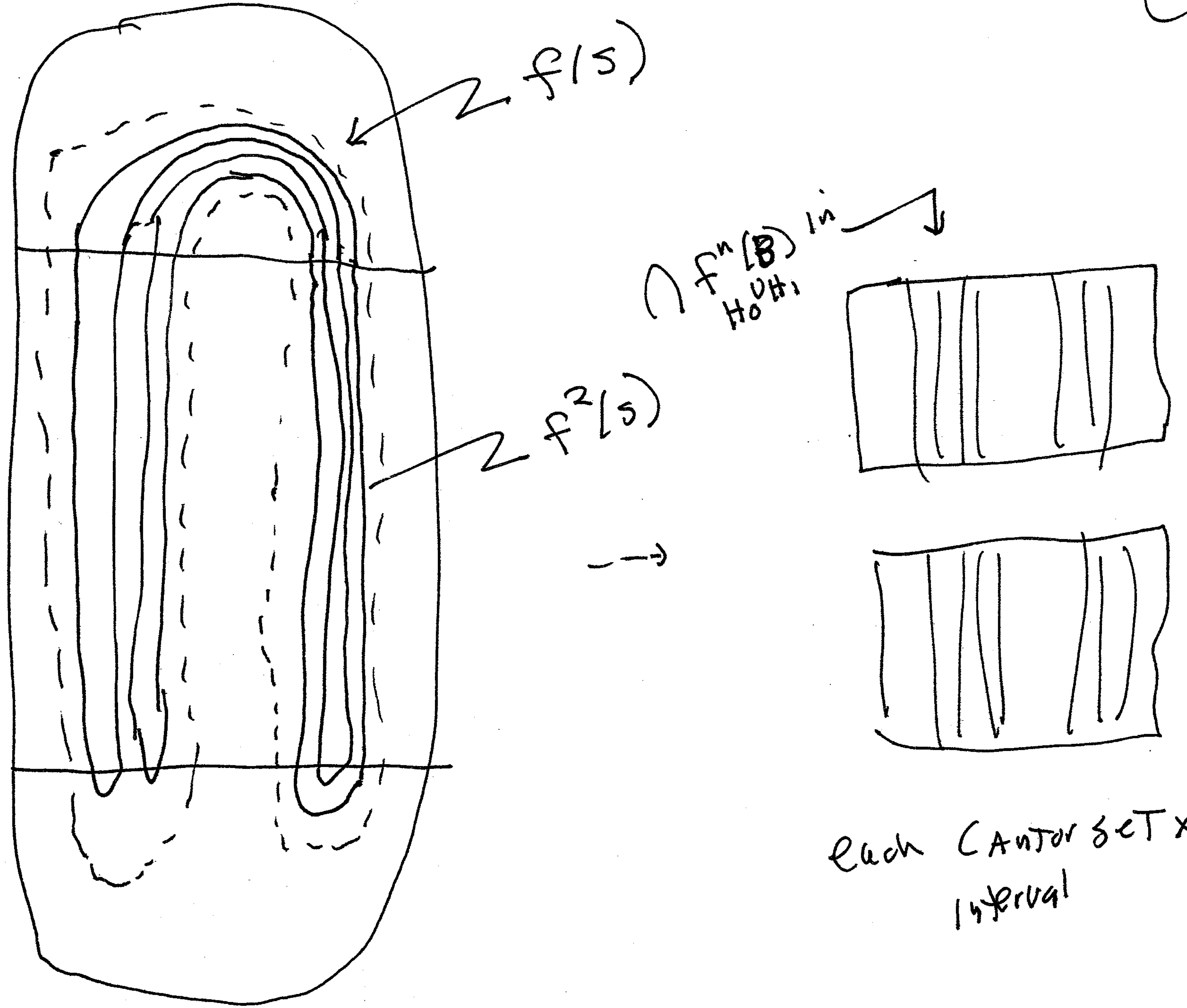
$\Lambda$

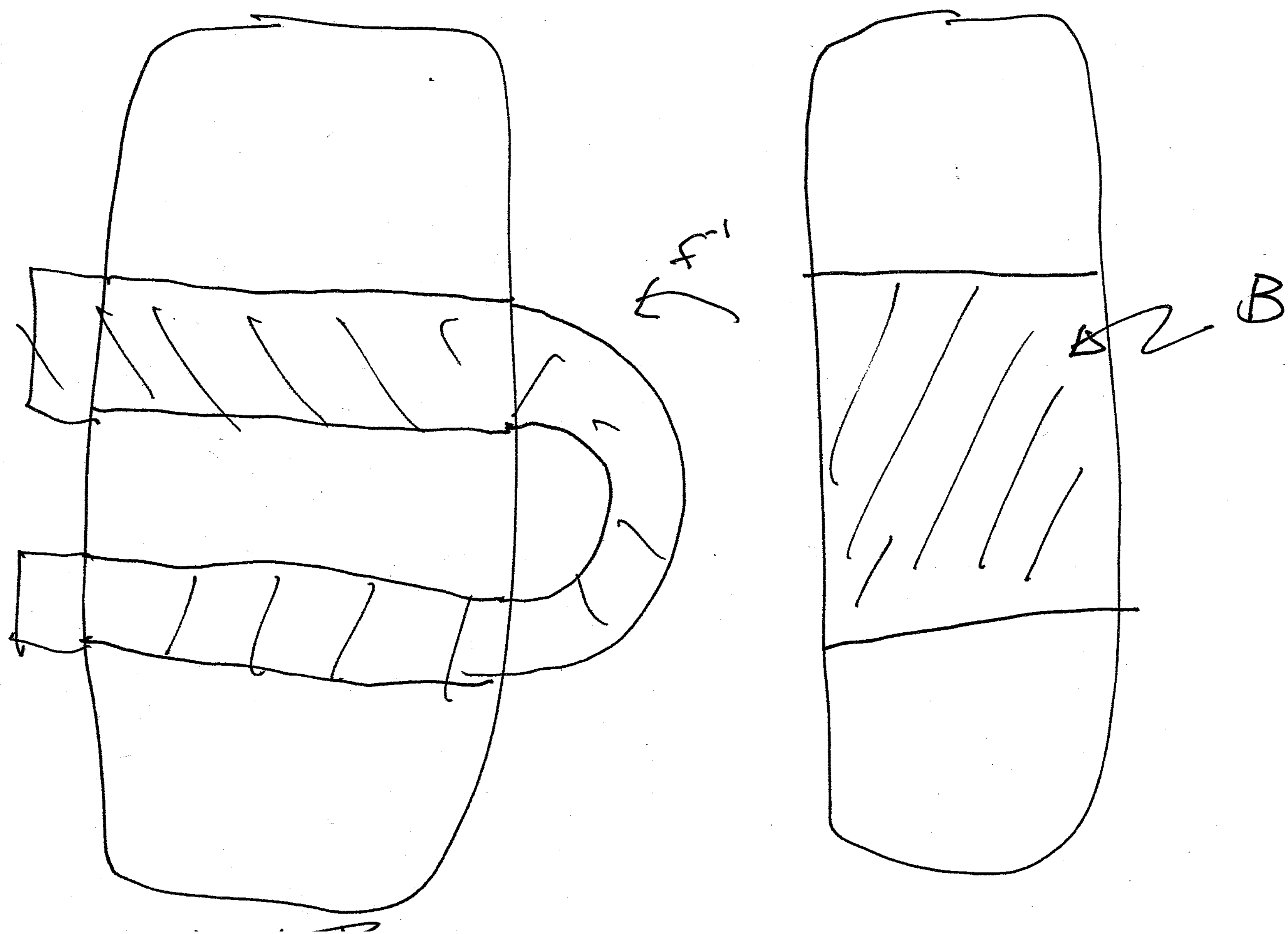
$\downarrow$

$\bullet\ P_+$

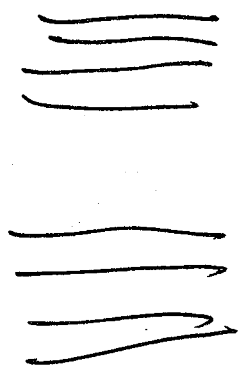
- We just sketch the details – See Devaney, "Intro to Chaotic Dyn. Sys"
Robinson "Stability, Symbolic Dynamics and Choas"


B
$H_b$
$H_1$
$G$
$H_0$
$H_a$
A
$B = H_1 \cup G \cup H_0$
The Box
S
$p_-$
Linear
Linear
$f(S)$
$f(G)$
$f(H_0)$
$f(H_1)$
$f(A)$
$p_+$
$f(B)$
Horseshoe
$p_-$


f(S)
f^2(S)
∩ f^n(B) in H_0 ∪ H_1
each Cantor set x interval

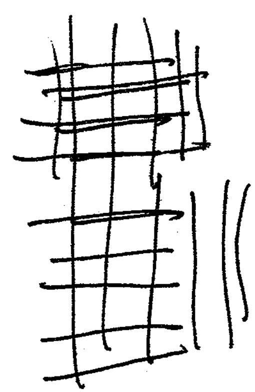

$f^{-1}$
B

In $\quad H_0 \cup H_1 \quad \bigcap_{n \in \mathbb{N}} f^{-n}(\mathbb{B}) = \text{Cantor set} \times \text{Interval}$



$$\Lambda = \bigcap_{k \in \mathbb{Z}} f^{n}(\mathbb{B}) = \text{Cantor set}$$

$$= \{x : f^{k}(x) \in \mathbb{B} \ \forall k \in \mathbb{Z}\}$$



$\boxed{\text{Play videos}}$

Use $H_0$ and $H_1$ as an address system for $\Lambda$

$i : \Lambda \to \Sigma_2 \quad (i(x))_k = s_k \iff f^k(x) \in H_{s_k}$

usual itinerary code

**Theorem:** $i : \Lambda \to \Sigma_2$ is a topological conjugacy.

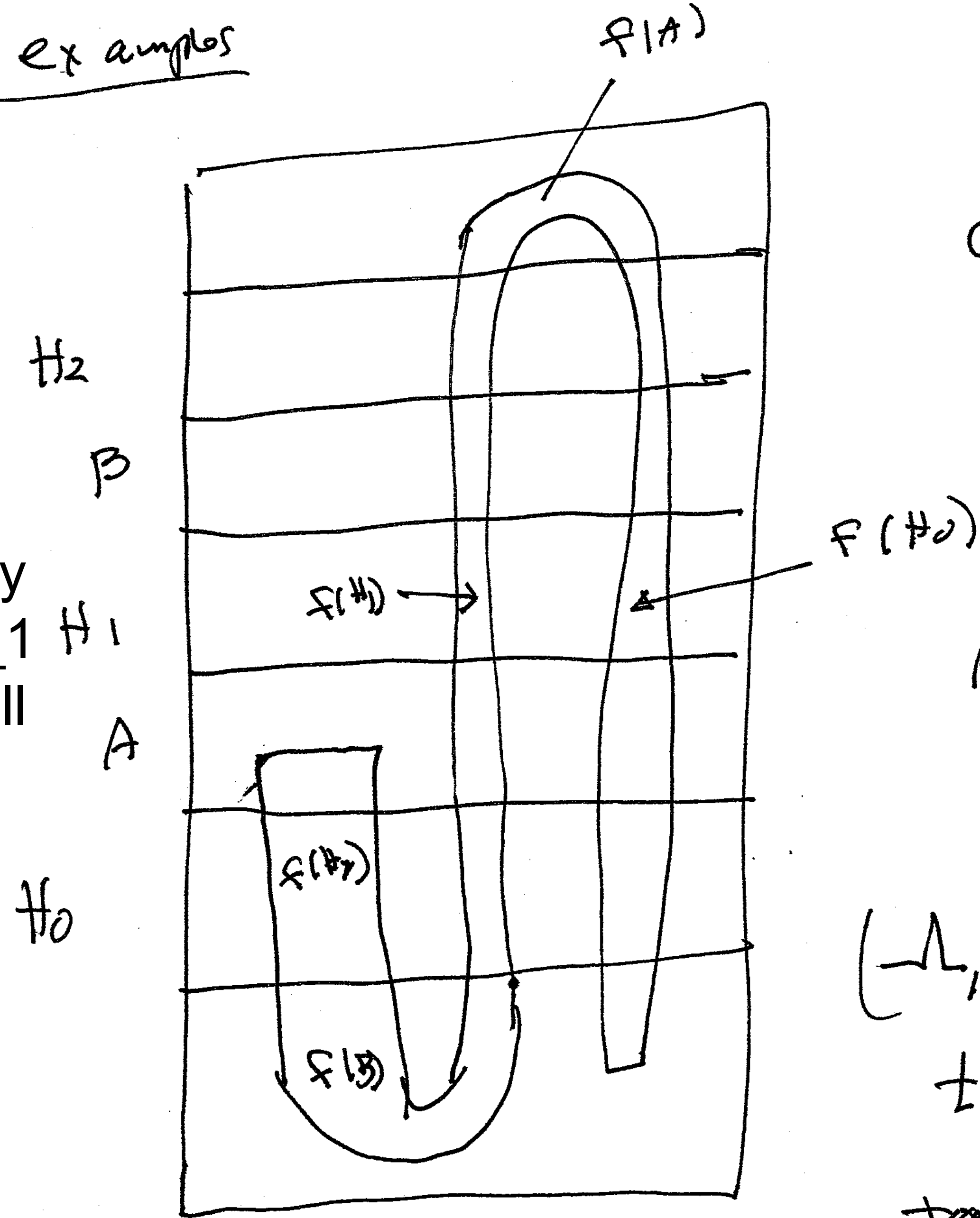
Much more to be said

- hyperbolic or $\Lambda$
- stable and unstable sets

$W^s(y) = \{x : f^n(x) \to y \text{ as } n \to \infty\}$

9

other examples




\Lambda is all points that stay in H_0\cup H_1 \cup H_2 for all forward and backwards iterates

$$\begin{tikzcd} & 1 \arrow[loop right] \arrow[dl] \arrow[dr] & \\ 0 \arrow[ur, bend left] \arrow[rr, bend left] & & 2 \arrow[ll, bend left] \end{tikzcd}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$(\Lambda, f|_\Lambda)$ top. conjugate to $(\Sigma_A, \sigma)$

top mixing $\Rightarrow$ transitive

$A^2 > 0$ so primitive

This lecture included these two youtube videos made by Prof. R. Ghrist at the University of Pennsylvania

https://www.youtube.com/watch?v=SrJm6bkLuPs

https://www.youtube.com/watch?v=skvCUST4LPk