DS13

1

- A common way to measure the complexity of a dynamical system is via growth rates of various dynamical quantities, eg $\#(\mathrm{Fix}(f^n))$

- Since dynamics is repeated iteration, the most common growth rates are exponential, λ^n, [repeated multiplication] and linear, $n\lambda$, repeated addition

- If $a_n > 0$ is a sequence, the exponential growth rate (egr) is

$$\mathrm{egr}(a_n) = \limsup_{n \to \infty} \frac{\log(a_n)}{n}$$

- So, for example, $a_n = \lambda^n$, $\mathrm{egr}(a_n) = \lim \frac{n \log \lambda}{n}$ $= \log(\lambda)$

- SAY $egr(a_n)$ converges, we can also compute

$$L(a_n) = \limsup_{n \to \infty} \sqrt[n]{a_n}$$

- So $a_n = c\lambda^n \Rightarrow L = \lambda$
- $e^{egr(a_n)} = L(a_n)$
- Now let Λ be a subshift of Σ_n^+ or Σ_n (i.e. compact fwd invariant or completely invariant subset)
- Let W_n = # distinct words (blocks) of length n in Λ
- The complexity of Λ as a "language" is

$$\mathrm{compl}(\Lambda) = egr(W_n(\Lambda)) = \limsup_{n \to \infty} \frac{\log W_n}{n}$$

- There are other names for the complexity and we will see that in terms of dynamics, complexity (Λ) is the topological entropy of $(\Lambda, \sigma|_\Lambda)$.

- What is the complexity of a SSFT Σ_A?

- Recall we showed that the number of paths of length n from i to j = number of allowable words starting with i and ending with j is

$$(A^n)_{ij}$$

- Thus $W_n = \sum_{i=1}^{n} \sum_{j=1}^{n} (A^n)_{ij} := \#(A^n)$

- So $\mathrm{Comp}(\Sigma_A) = \mathrm{egr}(W_n) = \mathrm{egr}(\#(A^n))$

- This is a linear algebra problem which is solved using the fundamental result about irreducible and primitive matrices. We give the primitive RESULT

<u>Perron-Frobenius</u>: Assume A is primitive $(\exists N, A^N > 0)$

(1) $\exists$ real, positive λ_1 that is simple root of the characteristic polynomial and $\lambda_1 > |\lambda|$ for any other eigenvalue λ.

(2) λ_1 has an eigenvector $\vec{V}^{(1)}$ with all entries positive and the eigenvectors for any other λ have positive and negative entries.

(3) Assume $\vec{x}$ is any vector with all components ≥ 0

then $\displaystyle \lim_{j \to \infty} \frac{A^j(\vec{x})}{|A^j(\vec{x})|} = \frac{\vec{V}^{(1)}}{|\vec{V}^{(1)}|}$ and

$\exists\ c_1, c_2 > 0$ so that

$$c_1 \lambda_1^j \leq |A^j \vec{x}| \leq c_2 \lambda_1^j$$

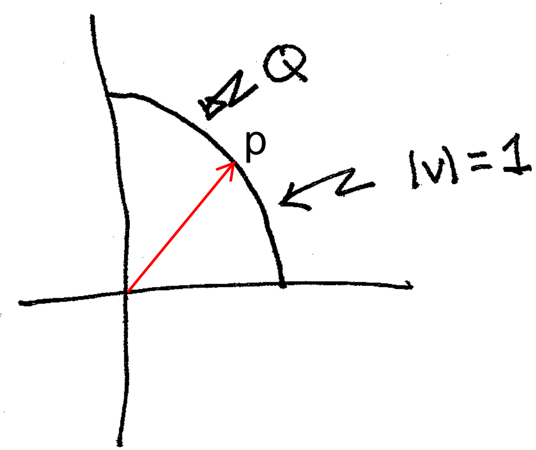
$$c_1 \lambda_1^j \leq (A^j \vec{x})_i \leq c_2 \lambda_1^j \qquad i = 1, .., n$$

$j > N$ (where $A^N > 0$).

For full proof see Gantmacher, "The Theory of Matrices"

6A

Ideas in proof



$\omega \in \{|v| = 1\} = S$

$\Phi(\omega) = \dfrac{A\omega}{|A\omega|}$, assume $A > 0$

then if $Q = S \wedge \{x_i \geq 0\}$

$\Phi : Q \to Q$ and is contracting in the correct metric $\Rightarrow \exists$ unique fixed point p i.e. $\dfrac{Ap}{|Ap|} = p$

so $Ap = \lambda p \quad \lambda > 0$

and assume eigenvectors form a basis

$\vec{x} = \alpha_1 V^{(1)} + \alpha_2 V^{(2)} + \ldots + \alpha_n V^{(n)}$

$A^n(x) = \alpha_1 \lambda_1^n V^{(1)} + \alpha_2 \lambda_2^n V^{(2)} + \ldots + \alpha_n \lambda_n^n V^{(n)}$

$$\frac{A^n(x)}{\lambda_1^n} = \alpha_1 \vec{V}^{(1)} + \alpha_2 \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)^n}_{\to 0} V^{(2)} + \ldots \alpha_n \underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)^n}_{\to 0} V^{(n)}$$

6B

OR

$$\frac{A^n(x)}{\|A^n x\|} = \frac{\alpha_1 \lambda_1^n \vec{V}_1 + \dots + \alpha_n \lambda_n \vec{V}_n}{\|\alpha_1 \lambda_1^n \vec{V}_1 + \dots + \alpha_n \lambda_n \vec{V}_n\|}$$

$$= \frac{\alpha_1 \vec{V}_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \dots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^n}{\|\alpha_1 \vec{V}_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \dots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^n\|}$$

$\left(\frac{\lambda_k}{\lambda_1}\right)^n \to 0$ since $\lambda_1 > |\lambda_k|$

and also $\lambda_1 > 0$.

<u>Theorem</u>: If A is primitive then

$$\text{comp}(\Sigma_A) = \log(\rho(A)), \quad \rho(A) = \text{spectral radius of } A = \lambda_1$$

<u>Proof</u>: We know that $W_n = \sum_{i=1}^{n} \sum_{j=1}^{n} (A^n)_{ij}$ = #A. Assume for the remainder of the proof that $(A^N) > 0$. Note that if $B > 0$ and $\vec{e} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow (B\vec{e})_i = B_{i1} + \dots + B_{in}$

= sum of the elements in the i^{th} row. Thus

$$\# B = \sum_{i=1}^{n} (B\vec{e})_i$$

Now, if $k > N$ we know from PF $\exists\, c_1, c_2 > 0$ with

$$c_1 \lambda_1^k \le (A^k \vec{e})_i \le c_2 \lambda_1^k$$

or summing

$$n c_1 \lambda_1^k \le \# A^k \le n c_2 \lambda_1^k$$

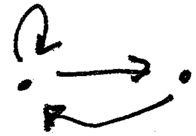
passing to the limit using $\#(A^k) = W_k$, we are done ∎

taking the log and dividing by k and

examples

- $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, full two shift, comp = $\log 2$

  and directly $W_k = 2^k$

- $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$  $\lambda = \frac{1 \pm \sqrt{1 - -4}}{2}$

  $\frac{1+\sqrt{5}}{2} = \rho(A)$

  $\text{comp}(\Sigma_A) = \log\left(\frac{1+\sqrt{5}}{2}\right)$

- Substitutions yield new examples of minimal sets Λ in Σ_n^+ which have $\text{compl}(\Lambda) = 0$ $\boxed{9}$

- Example $S: 0 \mapsto 01$ $S: 1 \mapsto 10$ are substitutions

  $0 \to 01 \to 0110 \to 01101001 \to 01101001 \vdots 10010110 \to \cdots$

  $\to \cdots = S^\infty(0) = \underline{m}$, the Morse sequence

  - if for a word w we let $S(w)$ be S performed on each symbol $S(s_0 s_1 \ldots s_n) = S(s_0) S(s_1) \ldots S(s_n)$

  - notice $S^{n+1}(0) = S^n(0)\overline{S^n(0)}$ where overbar is the substitution $0 \to 1$ $1 \to 0$

  - S acts on infinite sequences as well

  - $S(\underline{m}) = \underline{m}$, fixed point.

- Let $\Lambda = \mathrm{cl}(O^+(\underline{m}, \sigma))$ the orbit closure of $\underline{m}$ in the shift
- Facts: (Λ, σ) is minimal with $\mathrm{comp}(\Lambda) = 0$
- Notice two dynamics happens:

  S = substitution
  σ = shift

- S creates a fixed point $\underline{m}$ and it generates $\Lambda = \mathrm{cl}(O^+(\underline{m}, \sigma))$ which is minimal under the shift σ.