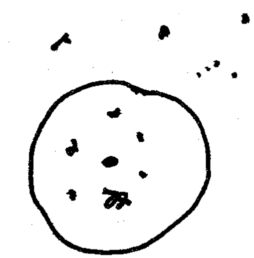
DS/4 (1)

- Recall x is (forward) recurrent if there exists $n_i \to \infty$ with $f^{n_i}(x) \to x$

- In many circumstances it is useful to understand more about the rate and structure of recurrence



- Given an open set U, $\mathrm{Return}(x, U, f) = \{ n \in \mathbb{N} : f^n(x) \in U \}$

- So, for example, if x is recurrent and $\varepsilon > 0$ $\mathrm{Return}(x, B_\varepsilon(x), f) \subseteq \mathbb{N}$ is unbounded i.e. infinite.

- A subset $A \subseteq \mathbb{N}$ is called <u>relatively dense</u> (or <u>syndetic</u>) if it has bounded gaps i.e. $\exists \underline{K}$ so that $\forall n$, $\{n, n+1, \ldots, n+K\} \cap A \neq \emptyset$
- eg $A = \{5k+3 : k \in \mathbb{N}\}$ has $K = 5$
- If $\mathrm{Return}(x, U, f)$ is relatively dense it means that $\forall n$, $\exists\ 0 \leq k \leq K$ with $f^{n+k}(x) \in U$
- A point x is called <u>almost periodic</u> if $\forall \varepsilon > 0$ $\mathrm{Return}(x, B_\varepsilon(x), f)$ is relatively dense
- eg $R_w : S^1 \circlearrowleft$ $w \notin \mathbb{Q}$, every point is almost periodic

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This Theorem of Birkhoff is frequently useful in proving that an orbit closure $\overline{O^+(x,f)}$ is minimal (i.e. every fwd orbit is dense).

<u>Theorem</u> Assume $f: X \to X$ is continuous, onto and X is compact metric
$\overline{O(x,f)}$ is minimal for $f \iff x$ is almost periodic.

<u>Proof</u>: ($\Leftarrow$) Assume x is almost periodic.
We need to show that every orbit is dense (orbit here means fwd orbit). Thus if $y \in \overline{O^+(x,f)}$ then we need $\overline{O^+(y,f)} = \overline{O^+(x,f)}$. For this it suffices to have $x \in \overline{O^+(y,f)}$ for then since $\overline{O^+(y,f)}$ is invariant $O^+(x,f) \subseteq \overline{O^+(y,f)}$ so $\overline{O^+(x,f)} \subseteq \overline{O^+(y,f)}$ and $\overline{O^+(y,f)} \subseteq \overline{O^+(x,f)}$ follows from $y \in \overline{O^+(x,f)}$

So $\forall\, y \in \overline{O^+(x,f)}$ we need $x \in \overline{O^+(y,f)}$ or

Given $\varepsilon > 0$, we must find an m with $f^m(y) \in B_\varepsilon(x)$.

Let $U = B_{\varepsilon/2}(x)$ and let K be such that $\forall n\ \exists k$ $0 \le k \le K$ with $f^{n+k}(x) \in U$

Now since X is compact metric, each f^k is uniformly continuous and so for $k = 1, \ldots, K$ let $\delta_k > 0$ be such that $d(x_1, x_2) < \delta_k \Rightarrow d(f^k(x_1), f^k(x_2)) < \frac{\varepsilon}{2}$

and let $\delta = \text{minimum } \delta_k > 0$

Since $y \in \overline{O^+(x,f)}$ $\exists\, n$ with $d(f^n(x), y) < \delta$

and let $0 \le k \le K$ be such that $f^{n+k}(x) \in B_{\varepsilon/2}(x)$

Thus $d(f^{n+k}(x), f^k(y)) < \varepsilon/2$ and so

$d(f^k(y), x) < \varepsilon$ as required.

For the converse we prove the negation. So assume x is not almost periodic. Thus $\exists\ B_\varepsilon(x) = U$ so that $\mathrm{Return}(x, B_\varepsilon(x), f)$ is not relatively dense.

Thus $\exists\ a_i \in \mathbb{N}$, $k_i \in \mathbb{N}$, $k_i \to \infty$ such that $a_i \to \infty$

$f^{a_i + j}(x) \notin U$ $0 \le j \le k_i$. Passing to a subsequence if neccessy using compactness assume $f^{a_i}(x) \to y$

Now fix j. By continuity $f^{a_i+j}(x) \to f^j(y)$ as $i \to \infty$. But for large enough i, $f^{a_i+j}(x) \notin B_\varepsilon(x) = U$ for all j and by construction. Thus $f^j(y) \notin B_\varepsilon(x)$

So $x \notin \overline{O^+(y,f)}$ so $\overline{O^+(y,f)}$ is not dense, and $y \in \overline{O^+(x,f)}$ so $\overline{O^+(x,f)}$ is not minimal.

$\blacksquare$

$\boxed{\text{BACK TO Substitutions more formally}}$

$A_n = \{0, \ldots, n-1\}$ the "alphabet"

$A_n^* =$ all finite words in the alphabet

A substitution is a map $S : A_n \to A_n^*$

eg $S(0) = 01 \quad S(1) = 10, \quad n = 2$

A substitution induces a map (with same symbol)

$$S(a_0 a_1 \ldots a_k) = S(a_0) S(a_1) \ldots S(a_k)$$

$$A_n^* \to A_n^*$$

and $S : \Sigma_n^+ \to \Sigma_n^+$, $S(.a_0 a_1 \ldots) = .S(a_0) S(a_1) \ldots$

DEF: If w_k is a sequence of words with $|w_k| \to \infty$ ($|w_k|$ = length of words) we say $w_k \to \underline{t} \in \Sigma_n^+$ if for some (and thus all) one sided sequences $\underline{b}$ the concatenation $w_k \underline{b} \to \underline{t}$ in Σ_2^+

- eg: $w_1 = 01$ $w_2 = 0101$ .. $w_k = (01)^k \Rightarrow w_k \to (01)^\infty$.

- The following simple criterion is usually used

  Assume $|w_k| \to \infty$ and for each k, w_k is a prefix of w_{k+1} $\Rightarrow \exists\, \underline{t}$ with $w_k \to \underline{t}$

where w is a <u>prefix of</u> v if $w_i = v_i$ for $i = 0, .., |w|-1$

- **FACT (HW)**: If S is a substitution with $|S^k(b)| \to \infty$ for all k, and $\exists\, a$ with $(S(a))_0 = a \Rightarrow \exists\, \underline{x}$ with $S^k(a) \to \underline{x}$ and $S(\underline{x}) = \underline{x}$ [fixed point]

- eg $S(0) = 01$ $S(1) = 10$

  $S^n(0) \to \underline{m}$ $S^n(1) \to \underline{m}'$ each fixed points of S

- eg $S(0) = 1$ $S(1) = 2$ $S(2) = 0$

  $0 \to 1 \to 2 \to 0 \to 1 \ldots$ so $|S^n(b)| \not\to 0$

  but 0^∞, 1^∞ and 2^∞ are all period 3 points of $S : \Sigma_2^+ \to \Sigma_2^+$

<u>The main object</u>: Assume now $S(\underline{a}) = \underline{a}$, $\underline{a} \in \Sigma_2^+$

let $\Lambda(\underline{a}) = \overline{O^+(\underline{a}, \sigma)}$ $\subset$ under the shift.

· When is $\Lambda(\underline{a})$ minimal ?

<u>DEF</u>: S is called <u>primitive</u>

if $\exists$ k such that $S^k(a)$ contains b

for all $a, b \in A_n$.

<u>eg</u> (1) $S(0) = 01$ $S(1) = 10$ is primitive with $k = 1$

(2) $S(0) = 1$, $S(1) = 2$, $S(2) = 0$ is not primitive

Tribonacci Substitution

$s(1) = 12 \qquad s(2) = 13 \qquad s(3) = 1$

$$1 \to 12 \to 1213 \to 1213121$$
$$2 \to 13 \to 121 \to 121312$$
$$3 \to 1 \to 12 \to 1213$$

$\boxed{k = 3}$ $\Rightarrow$ primitive

$s^{\infty}(1)$ is a fixed point

Theorem: Assume S is primitive and $(S(a))_0 = a \Rightarrow \Lambda(s^{\infty}(a)) = \overline{O(s^{\infty}(a), \sigma)}$ is a minimal set

<u>Proof</u>: Let $\underline{t} = S^{\infty}(a)$. Using the Theorem from the beginning of the lecture we must show $\forall \varepsilon > 0\ \exists K$ so that $\sigma^i(\underline{t}) \in B_\varepsilon(\underline{t})$ at least every K iterates i.e. $\exists K$ so that any list $\sigma^i(\underline{t}), \ldots, \sigma^{i+w}(\underline{t})$ with each not in $B_\varepsilon(\underline{t})$ has length $w < K$.

Now recalling that $d(\underline{s}, \underline{t}) < \varepsilon$ is equivalent to $s_i = t_i$ for $i = 0$ up to some N. we see that the condition in the last paragraph is equivalent to this: for any initial word W of $\underline{t}$, W occurs infinitely often in $\underline{t}$ with a uniform bound between the occurrences.

Now also recall that $\underline{t} = S^{\infty}(a) = \lim_{k\to\infty} S^k(a)$

Thus for any initial word W of $\underline{t}$ there is a m with W a prefix of $S^m(a)$ Therefore

- To prove minimality we must show that for all m, $S^m(a)$ occurs in $\underline{x}$ infinitely often with a uniform bound between occurrences.

- We first prove the case $m = 1$. Let K be as given in primitive so $S^K(b)$ contains c for all symbols b and c.

- For each pair of symbols b, c let M(b,c) be the minimum distance between occurrences of a in $S^K(b)\,S^K(c)$ and

$$M = \max\{M(b,c) : b, c \in A_n\}$$

since it is fixed.

Now if $\underline{x} = .a_0 a_1 a_2 \ldots$

$$S^K(\underline{x}) = \underline{x} = .S^K(a_0)\,S^K(a_1)\,S^K(a_2)\ldots$$

and so $\mathbf{a}$ occurs infinitely often in $\underline{x}$ with gaps bounded by M.

To prove the result for general m we do it informally at first

$$\underline{t} = a \overset{<M}{\longleftrightarrow} a \overset{<M}{\longleftrightarrow} a \ldots a \ldots a$$

$$t = S^m(\underline{t}) = S^m(a) \ldots S^m(a) \ldots S^m(a) \ldots$$

So if N is the maximum length of $S^m(W)$ where W is a word of length M, we see that $S^m(a)$ occurs in $\underline{t}$ with gaps bounded by N. $\blacksquare$