Preliminary remark

- We have seen that in Σ_n^+, $\underline{s}$ is close to $\underline{t}$ if $s_i = t_i$ for $0 \le i \le N$ for a large N.
- So in this lecture, instead of saying things like "given $\varepsilon > 0$ and $d(\underline{s}, \underline{t}) < \varepsilon$" we will say "given N with $s_i = t_i$ for $0 \le i \le N$" without explicitly mentioning the connection to $d(\underline{s}, \underline{t}) < \varepsilon$

- We want to examine the "language complexity" of substitution minimal sets

- Last time we showed

<u>Theorem</u>: If S is primitive and $(S(a))_0 = a$ then $\underline{x} = \lim_{m \to \infty} S^m(a)$ is a S fixed point and

$$\Omega(\underline{x}) = \overline{O^+(\underline{x}, \sigma)}$$ is a minimal set.

- Recall $W_m(\Omega)$ = # distinct words of length m in Ω.

- observation $W_m(\underline{x}) = W_m(\overline{O^+(\underline{x}, \sigma)})$

i.e. # of distinct words of length m in $\underline{x}$ is the same as those in $\overline{O^+(\underline{x}, \sigma)}$ in fact a word is in one $\Leftrightarrow$ it is in the other

The main theorem is that substitution minimal sets have linear complexity so comp$(\Lambda) = \text{egr}(W_n) = 0$

Theorem: Assume S is primitive, $(S(a))_0 = a$, $\underline{t} = S^{\infty}(a)$, $\Lambda = \overline{O^+(\underline{t}, T)}$ $\Rightarrow \exists C$ so that $W_m \leq Cm$, $\forall m \in \mathbb{N}$

- The proof needs a bit of machinery
- For a letter i and word w, $N_i(w) = \#$ of occurrences of i in w

$$\vec{N}(w) = \begin{bmatrix} N_1(w) \\ \vdots \\ N_n(w) \end{bmatrix} \quad \text{eg } w = 23113 \quad \vec{N} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

For the substitution its composition matrix is $M(S)$ defined by

$$(M(S))_{ij} = N_i(S(j)) = \#\text{ times } i \text{ occurs in } S(j)$$

so $\vec{N}(S(j))$ is the j^{th} column of M

eg $1 \to 12,\ 2 \to 13,\ 3 \to 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Properties
  (1) $M(S^n) = (M(S))^n$
  (2) $\vec{N}(S(w)) = M(S)\vec{N}(w)$
  (3) S is primitive $\Leftrightarrow$ M is primitive
  i.e. $\exists k$ with $M^k > 0$.

Proof: Let $\alpha_k = \min_{a \in A_n} |S^k(a)|$

$$\beta_k = \max_{a \in A_n} |S^k(a)|$$

Note that α_k and β_k are the min. and max. column sums of M^k

Now α_k is nondecreasing and since A is primitive $\alpha_k \to \infty$. Thus $\forall m \; \exists k = k(m)$ with

$$\alpha_{k-1} \leq m \leq \alpha_k$$

(1) Every word of length m is in $S^k(ab)$ for some pair of letters ab in Σ

(2) $\exists$ at most β_k different words in $S^k(ab)$ with initial symbol in $S^k(a)$

These imply $W_m \le n^2 \beta_k$ (Recall $n = \#$ of symbols)

Now recalling that α_k and β_k are column sums of M^k, by Perron-Fröbenius $\exists\, c_1, c_2 > 0$

with $c_1 \lambda_1^k \le \alpha_k \le \beta_k \le c_2 \lambda_2^k$

where λ_1 is the eigenvalue of max modulus of M.

We have then

$$W_m \le n^2 \beta_k \le n^2 c_2 \lambda_2^k = \left(\frac{c_2}{c_1} n^2 \lambda\right) c_1 \lambda^{k-1}$$

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$$\le \left(\frac{c_2}{c_1} n^2 \lambda\right) \alpha_{k-1} \le \left(\frac{c_2}{c_1} n^2 \lambda\right) m.$$

↑ Constant C.

The Adding machine (a bit informally).

$\Sigma_2^+ = \{0,1\}^{\mathbb{N}} = \bigoplus \mathbb{Z}_2$ we give it the structure of a group by adding with carrying

$$\begin{array}{r} .0110\ldots \\ +\ .1000\ldots \\ \hline .1110\ldots \end{array}$$

$$\begin{array}{r} .1100\ldots \\ +\ .100\ldots \\ \hline 0010\ldots \end{array}$$

Define $A: \Sigma_2^+ \to \Sigma_2^+$ as $A(\underline{s}) = \underline{s} + .1000\ldots$

$\Rightarrow$ A is a homeomorphism

Theorem (Σ_2^+, A) is minimal

Proof 1st we claim that if $\underline{t} = 0^\infty$

$\Rightarrow \Sigma_2^+ = \overline{O^+(\underline{t}, A)}$. For this we

fix $\underline{s} \in \Sigma_2^+$ and N and we must show $\exists m$

with $(A^m(\underline{t}))_i = s_i$ for $i = 0, \dots, N$.

Treat $.s_0 s_1 \dots s_N$ as the base 2 number

$m =$ S_N .... S_1 S_0 $= s_0 + s_1 \cdot 2 + s_2 \cdot 2^2 + \dots + s_N 2^N$

then $A^m(0^\infty) = .s_0 s_1 \dots s_N 0^\infty$

so $s_0 s_1 \dots s_N = 0 + 1 \cdot m$ times $= 0 + m \cdot 1$

Now we show $\underline{t} = 0^\infty$ is almost periodic

So given N we first produce m so that

$$A^m(0^\infty)_i = 0 \quad \text{for } i = 0, \dots, N$$

Using the base 2 rep, let $m = 2^{N+1}$

then $$A^{2^{N+1}}(0^\infty) = .0 \dots 0.1 0$$
↑ N+1 place.

Similarly, A^{km}(0^\infty} = 0 ...0*** for any k, so \underline{t} is almost periodic.

This is an example of a big source of minimal sets, namely, translations on compact, Abelian topological groups. (Kronecker systems) other examples

(1) +1 with carrying on

$\bigoplus \mathbb{Z}_{p_i}$ with p_i increasing sequence of primes

(2) S^1 the circle, $\theta \to \theta + \omega$ is translation on S^1 treated as additive group

(3) $\mathbb{T}^n = S^1 \times \cdots \times S^1$ with translation is sometimes minimal others not depending on translation amount