- We study the dynamics of linear maps $\mathbb{R}^n \to \mathbb{R}^n$

- applications
  - local dynamics near fixed or periodic points
  - Linear Toral automorphisms

- Recall Taylor expansion for $f : \mathbb{R}^n \to \mathbb{R}^n$ expanded about x_0

- $f(x_0 + x) = f(x_0) + Df(x_0)x + h.o.t$
  where $Df(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0)\right)$ is $n \times n$ derivative matrix

- Assume 0 is a fixed point then Taylor is

- $f(x) = Df(0)\,x + \text{h.o.t.}$
- Look at the linear part $f(x) = Ax + \text{h.o.t}$

  A is an $(n \times n)$-matrix
- **Question** How much does the dynamics near zero depend just on the matrix $A = Df(0)$
- **Answer**: In the "generic case" (to be specified) $\exists$ neighborhood U of zero where the dynamics of f "looks like" (is topologically conjugate to) that of $x \longmapsto Ax$ [Hartman-Grobman Theorem]
- So we need to understand dynamics of $L(x) = Ax$ - we stick to A is $n \times n$ so $L : \mathbb{R}^n \to \mathbb{R}^n$ can be iterated and also A^{-1} exists so L^{-1} does

- An ingredient is the Contraction Mapping Theorem which is a fully non-linear result often used to get existence results (used in function spaces) and convergence of numerical algorithms.

- DEF: $f:(X,d)\to(X,d)$ is called a <u>contraction</u> if $\exists\ 0<\lambda<1$ so that $d(f(x),f(x'))\le\lambda d(x,x')$ for all $x,x'\in X$. (so f is uniformly Lipschitz with constant λ)

- <u>Theorem</u>: Assume (X,d) is a complete metric <u>space</u> (eg X cpt. metric or $X=\mathbb{R}^n$) and $f:X\to X$ is a contraction $\Rightarrow$ f has a unique fixed point p and for all $x\in X$, $f^n(x)\to p$ as $n\to\infty$ or $p=\omega(x,f)$.

<u>Proof</u>: Pick $x \in X$, then $d(f^2(x), f(x)) \leq \lambda\, d(x, f(x))$ and by induction $d(f^{n+1}(x), f^n(x)) \leq \lambda^n d(x, f(x))$

- Now given $\varepsilon > 0$, let N be such that $n \geq N$ implies $\sum_{n=N}^{\infty} \lambda^n d(f(x), x) < \varepsilon$.

- If $m > n \geq N$

$$d(f^n(x), f^m(x)) \leq d(f^n(x), f^{n+1}(x)) + \cdots + d(f^{m-1}(x), f^m(x))$$

$$\leq \lambda^n d(f(x), x) + \cdots + \lambda^m d(f(x), x)$$

$$< \varepsilon$$

Thus $\{f^n(x)\}$ is a Cauchy sequence and so $\exists p$ with $f^n(x) \to p$ and so $f^{n+1}(x) \to f(p)$ Thus $p = f(p)$.

Now if $f(p) = p$ and $f(p') = p' \Rightarrow d(p, p') = d(f(p), f(p')) \leq \lambda d(p, p')$ so $d(p, p') = 0$ or $p = p'$ ∎

<u>example</u> $L(x) = Ax$, $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $0 < \lambda_1 \leq \lambda_2 < 1$

$d(x,y) = \|x-y\|_2$ (two norm)

$\Rightarrow d(Ax, Ay) = \|Ax - Ay\|_2 = \|A(x-y)\|_2 \leq \|A\|_2 \|x-y\|_2$

$= \lambda_2 d(x,y)$

So L is a contraction, and $A(\vec{0}) = \vec{0}$, it is the global attracting fixed point.

. This is obvious $L^n(x,y) = (\lambda_1^n x, \lambda_2^n y) \to 0$ as $n \to \infty$.

Basic Linear algebra

both $\neq 0$

$\exists$ A has eigenvalues $\lambda_1 \neq \lambda_2$ and $C = [\vec{v}_1 \ \vec{v}_2]$ with $\vec{v}_i$ the corresponding eigenvalues

then $C^{-1} A C = \mathrm{diag}(\lambda_1, \lambda_2)$

so $A = C \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} C^{-1} = C D C^{-1}$,

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "C^{-1}"'] & \mathbb{R}^2 \arrow[d, "C^{-1}"] \\ \mathbb{R}^2 \arrow[r, "D"] & \mathbb{R}^2 \end{tikzcd}$$

$C^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is homeo

Thus $L(\vec{x}) = A\vec{x}$ and $\bar{L}(x) = Dx$ are topologically conjugate and have the same dynamics.

<u>Thus</u> If A has eigenvalues $0 < \lambda_1 < \lambda_2 < 1$

$\Rightarrow$ $\vec{0}$ is the unique fixed point of $L(\vec{x}) = Ax$

and for any $\vec{x} \in \mathbb{R}^2$, $\omega(x, L) = \vec{0}$.

- The strategy for general $L(\vec{x}) = Ax$ is to find a simple model matrix M with $\bar{L}(x) = Mx$ having understandably dynamics and an invertible C with $A = CMC^{-1}$

- The matrix M is provided by the real Jordan form.

Real Jordan Form

Assume A is 2×2 with real entries and eigenvalues λ_1, λ_2 $\Rightarrow$ $\exists$ invertible real matrix C so that according to conditions on λ_i takes the form

$$M = C^{-1} A C$$

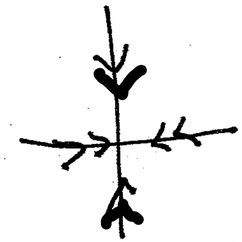
(A) $\lambda_1 \neq \lambda_2$ both real $\quad M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

(B) $\lambda_1 = \lambda_2$ $\quad M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$ or $M = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$

(C) $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\beta \neq 0$ $\quad M = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

Dynamics of model system.

(A) (1) $0 < \lambda_1 < \lambda_2 < 1$

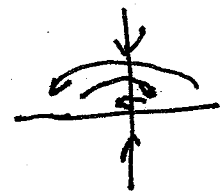


$x_n = x_0 \lambda_1^n$

$y_n = y_0 \lambda_2^n$

$\vec{0}$ is global attractor i.e. $\omega(\vec{x}) = 0 \ \forall x$

(2) $-1 < \lambda_1 < 0 < \lambda_2 < 1$



$x_n = x_0 \lambda_1^n$

$y_n = y_0 \lambda_2^n$

$\vec{0}$ is global attractor

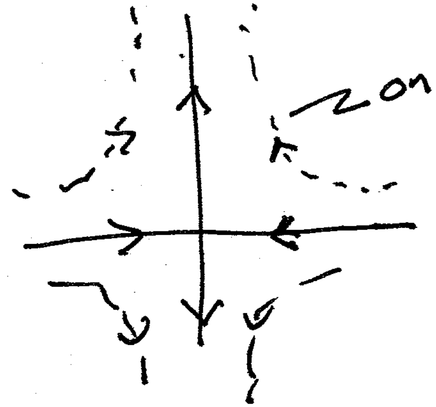
(3) $-1 < \lambda_1 < \lambda_2 < 0$

$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ times D so dynamics

like (1) but with flip in both coordinates.

$\vec{0}$ is a global attractor

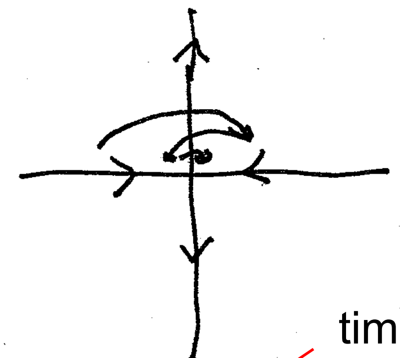
(4) $0 < \lambda_1 < 1 < \lambda_2$



on hyperbolas

Saddle point

(5) $-1 < \lambda_1 < 0 < 1 < \lambda_2$



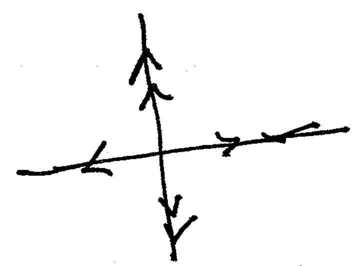
[degenerat cases $\lambda_1 = 1$]

(6) $\lambda_1 < -1 < \lambda_2 < 0$

$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ so (1) with a flip

times D

(7) $1 < \lambda_1 < \lambda_2$



(8) $\lambda_1 < -1 < 1 < \lambda_2$

like (7) with one flip

(9) $\lambda_1 < \lambda_2 < -1$

double flip

Source at $\vec{0}$, $\alpha(x) = \vec{0}$ $\forall x$

(c) $$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \sqrt{\alpha^2+\beta^2} \begin{bmatrix} \alpha/\sqrt{\alpha^2+\beta^2} & \frac{-\beta}{\sqrt{\alpha^2+\beta^2}} \\ \frac{\beta}{\sqrt{\alpha^2+\beta^2}} & \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \end{bmatrix}$$

$\lambda = \alpha \pm i\beta$

$$= |\lambda| \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = |\lambda| R_\theta$$

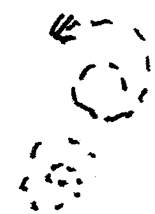
↑ rigid rotation by θ

$\theta = \operatorname{Arg}(\alpha + i\beta)$

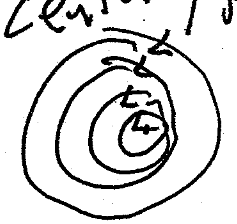
$$\boxed{M^n = |\lambda|^n R_{n\theta}}$$

(1) $|\lambda| > 1$ spiral out

(2) $|\lambda| < 1$ spiral in



(3) $|\lambda| = 1$ center, pure rotation

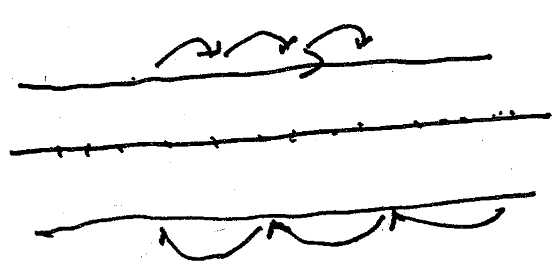


(B) $M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ $\lambda \neq 0$

we do a conjugation $C' = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, $C' M (C')^{-1}$

$= \begin{bmatrix} \lambda & \lambda \\ 0 & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(a) $\lambda = 1$



line of fixed pts (This is called a shear)

(b) $|\lambda| < 1$ Shear than shrink by λ

(c) $|\lambda| > 1$ Shear then ex ~~and~~ by λ

<u>Summary</u>: when all $|\lambda| < 1 \Rightarrow L^n(\vec{x}) \to \vec{0}$ all $\vec{x}$

when all $|\lambda| > 1 \Rightarrow |L^n(\vec{x})| \to \infty$ all $\vec{x} \neq 0$

<u>Theorem</u> A is 2×2 with eigen values λ_1, λ_2

if $|\lambda_1|, |\lambda_2| < 1 \Rightarrow \forall \vec{x} \in \mathbb{R}^2,\ A^n \vec{x} \to 0$