Summary: when all $|\lambda| < 1 \Rightarrow L^n(\vec{x}) \to \vec{0}$ all $\vec{x}$

when all $|\lambda| > 1 \Rightarrow |L^n(\vec{x})| \to \infty$ all $\vec{x} \neq 0$

L(x) = Ax

Theorem: A is 2×2 with eigen values λ_1, λ_2

if $|\lambda_1|, |\lambda_2| < 1 \Rightarrow \forall \vec{x} \in \mathbb{R}^2,\ A^n \vec{x} \to 0$

Proof: Using the real Jordan form it reduces to the three cases. We show in all cases $\forall \vec{x} \in \mathbb{R}^2$

$\exists\ 0 < \nu < 1$ so that $\Phi(A\vec{x}) \leq \nu\, \Phi(\vec{x})$

where $\Phi(\vec{x}) = x^2 + y^2 = \|\vec{x}\|^2$

Given this $\forall x$, $\Phi(A^n\vec{x}) \to 0$ as $\vec{n} \to \infty$

which implies $A^n \vec{x} \to 0$

we go case by case

(A) $M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow \Phi(M\vec{x}) = \| \lambda_1 x, \lambda_2 y \|^2 \leq$

$\max |\lambda_i|^2 \| (x,y) \| = \max |\lambda_i|^2 \Phi(\vec{x})$

Let $\nu = \max |\lambda_i|^2$

(C) $M = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = |\lambda| R_\theta$ now $\| R_\theta (\vec{x}) \| = \| \vec{x} \|$ since

it is orthogonal, so $\Phi(M\vec{x}) = \| |\lambda| R_\theta (\vec{x}) \|^2$

$= |\lambda|^2 \| \vec{x} \|^2 = |\lambda|^2 \Phi(\vec{x})$ so $\nu = |\lambda|^2$

(B) We need another conjugation trick

Let $C = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ for some $\varepsilon > 0$

$\Rightarrow C M C^{-1} = \begin{pmatrix} \lambda & \varepsilon \\ 0 & \tau \end{pmatrix} := M'$ Thus the dynamics of $M\vec{x}$ are the same as $M'\vec{x}$.

Now $\Phi(M'\vec{x}) = \left\| \begin{pmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2$

$$= (\lambda x + \varepsilon y)^2 + (\lambda y)^2$$

$$= \lambda^2 x^2 + \lambda^2 y^2 + \varepsilon^2 y^2 + 2\varepsilon\lambda x y$$

$$\le (\lambda^2 + \varepsilon^2)(x^2 + y^2) + \varepsilon\lambda 2 x y$$

added εx^2

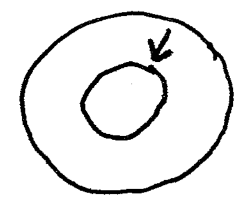
$2xy \le x^2 + y^2$

$$\le (\lambda^2 + \varepsilon^2 + \varepsilon|\lambda|)(x^2 + y^2)$$

$$= (\lambda^2 + \varepsilon^2 + \varepsilon|\lambda|)\, \Phi(\vec{x})$$

Since $|\lambda| < 1$ we may chose ε small enough

so that $\lambda^2 + \varepsilon^2 + \varepsilon|\lambda| < 1$ $\boxed{\varepsilon}$



Φ is "strong" Lyapunov function

Level sets go stricly inside themselves.

- As a point of clarification, for soln to linear DE $\frac{dx}{dt} = Ax$, A has e.values $\{\lambda_1, \dots, \lambda_n\}$

$$Re(\lambda) < 0 \Rightarrow x \text{ is global attractor}$$

- The informal relation is the soln to DE has terms like $e^{\lambda t}$ and so $Re(\lambda) < 0$

$$\Rightarrow e^{\lambda t} \to 0 \text{ as } t \to \infty.$$

- We will cover the DE case later.

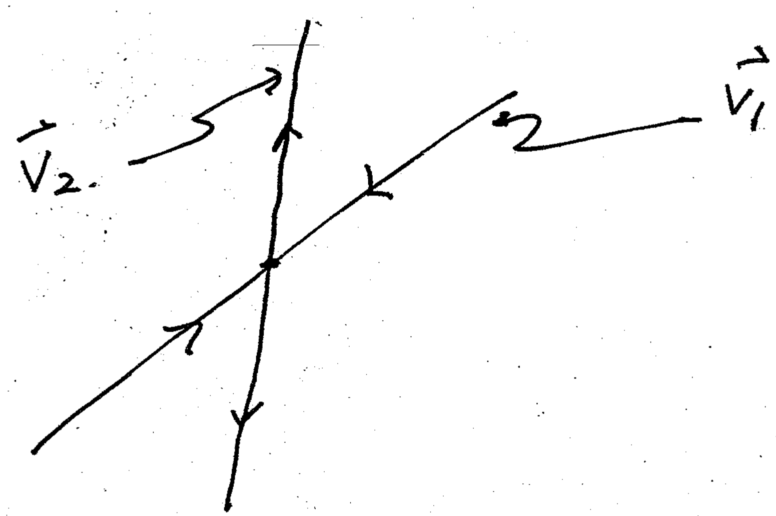
Remarks on $0 < \lambda_1 < 1 < \lambda_2$. Saddle.

If the eigenvectors are $\vec{v}_1$ and $\vec{v}_2$ which form a basis

If $\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$

$$A^n \vec{w} = \alpha_1 A^n \vec{v}_1 + \alpha_2 A^n \vec{v}_2$$
$$= \alpha_1 \lambda_1^n \vec{v}_1 + \alpha_2 \lambda_2^n \vec{v}_2$$

. So in eigen coordinates A decouples (equivalent to diagonalization)



The <u>stable manifold</u> of $\vec{0}$ is

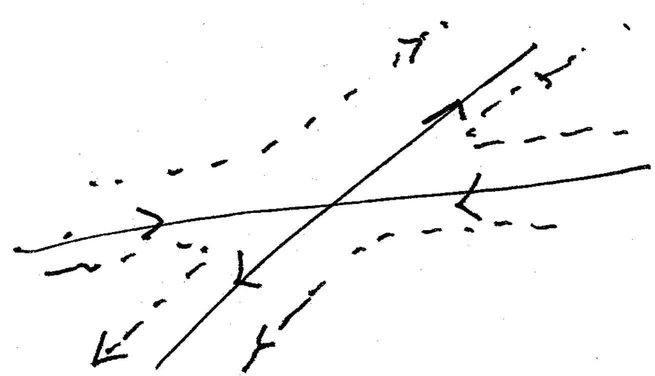
$$W^s(\vec{0}) = \{ x : A^n x \to \vec{0} \text{ as } n \to \infty \}$$

The <u>unstable manifold</u> of $\vec{0}$ is

$$W^u(\vec{0}) = \{ x : A^n x \to \vec{0} \text{ as } n \to -\infty \}$$

So $W^s(\vec{0}) =$ line in direction of $\vec{v}_1$ through the origin

$W^u(\vec{0}) =$ " " " $\vec{v}_2$ " "



- The higher dimensional theory is similar, but the real Jordan form is more complicated.

- <u>Theorem</u>: Given real $n \times n$ matrix A, $\exists$ an invertible matrix C so that $M = C^{-1}AC$ and M has the form:

$$M = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_k \end{bmatrix}$$

where each block B_i depending on the eigenvalue of A has one of the following forms

(1) $B_i = \lambda$ for some real eigenvalue λ

(2) $B_i = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$ for some real eigenvalue λ

(3) $B_L = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ where $\lambda = \alpha \pm i\beta$, $\beta \neq 0$ are eigenvalues of A

(4) $B_L = \begin{bmatrix} D & I & & 0 \\ & D & I & \\ & & \ddots & I \\ 0 & & & D \end{bmatrix}$ where $D = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

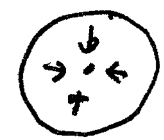
for some eigenvalue $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

Theorem: If $L : \mathbb{R}^n \to \mathbb{R}^n$ is $L\vec{x} = A\vec{x}$ and all eigenvalues λ of A have $|\lambda| < 1 \Rightarrow \forall \vec{x} \in \mathbb{R}^n$, $\lim_{k \to \infty} L^k(\vec{x}) = \vec{0}$, $\lim_{k \to -\infty} \|L^k(\vec{x})\| = \infty$ and $\vec{0}$ is the unique fixed point

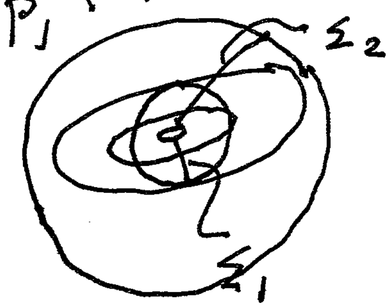
There is a more refined analysis of fixed points (WATCH OUT - Definitions vary in the literature).

Now $f: X \to X$, $f(p) = p$, not necessarily linear

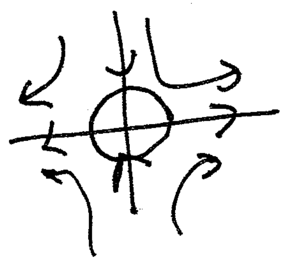
<u>DEF</u>: (1) p is called <u>aspmtotically stable</u> if $\exists\, \varepsilon > 0$ so that $d(x,p) < \varepsilon \Rightarrow f^n(x) \to p$ as $n \to \infty$

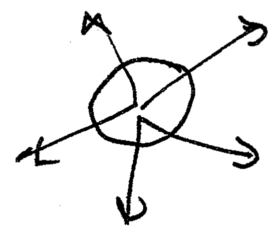


(2) p is called stable if $\exists\, 0 < \varepsilon_1 < \varepsilon_2$ so that $d(p,x) < \varepsilon_1 \Rightarrow d(p, f^n(x)) < \varepsilon_2$ $\forall n > 0$



(3) p is unstable if it is not stable





__Theorem__: $L: \mathbb{R}^n \to \mathbb{R}^n$ via $L(\vec{x}) = A\vec{x}$ and $\exists$ eigenvalue λ with $|\lambda| > 1 \Rightarrow \vec{0}$ is unstable

- The case where all $|\lambda| \leq 1$ can be tricky because of the off diagonals in the Real Jordan form.

DEF:

- Assume $f: \mathbb{R}^n \to \mathbb{R}^n$ is such that both f and f^{-1} (so f is invertible) have r-derivatives that are continuous, then f is called a C^r-diffeomorphism and one writes $f \in \mathrm{Diff}^r(\mathbb{R}^n)$

- <u>Hartman-Grobman for diffeomorphisms</u>:

Assume $f \in \mathrm{Diff}^2(\mathbb{R}^n)$ and $f(p) = p$, $A = Df(p)$

and no eigenvalue of A has modulus one!

Then $\exists$ neighborhoods U of p and V of $\vec{0}$ ~~with~~ and a homeomorphism $h: U \to V$ so that

$$\begin{array}{ccc} U_0 & \xrightarrow{f|_{U_0}} & U \\ \downarrow h & & \downarrow h \\ V_0 & \xrightarrow{L|_{V_0}} & V \end{array}$$

where $U_0 \subseteq U$

$V_0 \subseteq V$

where $L(\vec{x}) = A\vec{x}$

CORR: If $f \in \mathrm{Diff}^2(\mathbb{R}^n)$, $f(p) = p$ $A = Df(p)$

$\Rightarrow$ (1) If all eigenval λ of A are $|\lambda| < 1 \Rightarrow p$ is asymptotically stable

(2) If $\exists\, \lambda$ of A with $|\lambda| > 1 \Rightarrow p$ is unstable.

Example in next lecture

and application to periodic points