- A fixed point p of a C^1 $f:\mathbb{R}^n \to \mathbb{R}^n$ is called <u>hyperbolic</u> if all eigenvalues of $Df(p)$ do <u>not</u> have modulus 1 – or $\mathrm{spec}(Df(p)) \cap S^1 = \emptyset$

- This is standard but confusing. If all $|\lambda|<1$ it is an attractor which has no hyperbolas.

- A more natural def of hyperbolic is some $|\lambda|>1$ and some $|\lambda|<1$, but we use the standard terminology

- <u>HARTMAN Grobman</u>: If $f(p)=p$ and f is C^2 and p is hyperbolic then near p, f is conjugate to $x \mapsto Df(p)x$ i.e. its linearization. Thus the spectrum of $Df(p)$ decides whether p is attractor, repeller or saddle.

<u>Example</u>: Find the fixed points of

$F(x,y) = (-4-4y-x^2, x)$ and classify them.

<u>SOLN</u>: $(x,y) = F(x,y)$ for fixed points yields

$$-4-4y-x^2 = x$$
$$x = y$$

So $-4-4x-x^2 = x$

or $x^2+5x+4=0$

or $(x+4)(x+1)=0$

$x=-4 \quad y=-4$: P_1

$x=-1 \quad y=-1$: P_2

$$DF(x,y) = \begin{bmatrix} -2x & -4 \\ 1 & 0 \end{bmatrix}$$

$$DF(p_1) = \begin{bmatrix} 8 & -4 \\ 1 & 0 \end{bmatrix}$$

Recall for 2x2

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

T = trace
D = det

$T=8, D=4$: $\lambda = \frac{8 \pm \sqrt{64-16}}{2} = \frac{8 \pm \sqrt{48}}{2} = \frac{8 \pm 4\sqrt{3}}{2} = 4 \pm 2\sqrt{3}$

$\approx 4 \pm 3.46$

$= .54, 7.46$

one $|\lambda| < 1$ one $\lambda > 1$
so saddle.

$$DF(p_2) = \begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix}$$

$T=2, D=4$: $\lambda = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2\sqrt{3}\,i}{2} = 1 \pm \sqrt{3}\,i$

$|\lambda| = \sqrt{1+3} = 2$ so source, unstable spiral.

- A least period n periodic point p is called stable, asymptotically stable, unstable if that is what it is treated as a fixed point of f^n

- NOTE: By the chain Rule

$$D(f^n(p)) = Df(f^{n-1}(p)) \cdot Df(f^{n-2}(p)) \cdots Df(p)$$
$$= \text{product of derivatives along the orbit}$$

- it is often hard to see from the intermediate phases what the end result will be in terms of stability

- There are packages which find periodic points and compute their stability type

- Before moving on to flows and Diff Eq we use linear dynamics in a different way to study Toral automorphisms

- We need to Review a bit about quotient spaces and covering spaces. We only use what is needed here. For more info see, for example, Munkres or Hatcher.

- We stick to dimension 2 but everything generalizes to higher dimensions

- $\mathbb{R}^2$ = plane $\mathbb{Z}^2$ = integer lattice.

  $\mathbb{R}^2/\mathbb{Z}^2$ means the set of equivalence classes.

  $[(x,y)]$ where $(x,y) \sim (x',y') \iff$

  $\exists\, (m,n) \in \mathbb{Z}^2$ with $(x',y') = (x+m, y+n)$

- More compactly, $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ via

  $(m,n) \cdot (x,y) = x+m, y+n$

  and $\mathbb{R}^2/\mathbb{Z}^2$ is the space with $(x,y) \sim (x',y')$

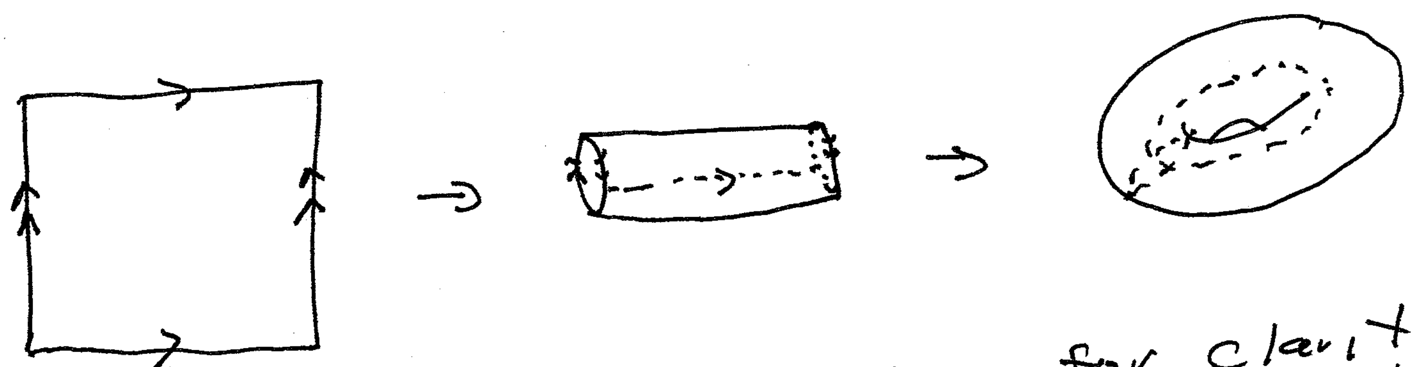
  $\iff$ they are on the same $\mathbb{Z}^2$-orbit

  $\iff (x',y') = g \cdot (x,y)$ for some $g \in \mathbb{Z}^2$

- So $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ the 2-dimensional torus

- We also think of $\mathbb{T}^2$ as obtained by identifying opposite sides of the unit square

- The unit square is a fundamental domain of the $\mathbb{Z}^2$ action



Shown in two steps for clarity

- When does a map $F: \mathbb{R}^2 \to \mathbb{R}^2$ induce a map $f: \mathbb{T}^2 \to \mathbb{T}^2$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R}^2 \arrow[d, "\pi"] \\ \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z} \arrow[r, "f"] & \mathbb{R}^2/\mathbb{Z} = \mathbb{T}^2 \end{tikzcd}$$

the projection

- To be well defined we need that

when $(x,y) \sim (x',y') \Rightarrow F(x,y) \sim F(x',y')$

- OR if $(x',y') = (x+m, y+n)$

$\Rightarrow F(x',y') = F(x,y) + (k,\ell)$ for $(k,\ell) \in \mathbb{Z}^2$

note that (k,ℓ) doesn't have to be the same as (m,n)

- Example, let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with all entries in $\mathbb{Z}$

$\Rightarrow L(x,y) = M\begin{pmatrix} x \\ y \end{pmatrix}$ descends since

$$L(x+m, y+n) = M\begin{pmatrix} x+m \\ y+n \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix} + M\begin{pmatrix} m \\ n \end{pmatrix}$$

$$= L(x,y) + \begin{pmatrix} am+bn \\ cm+dn \end{pmatrix}$$

- we want invertible maps so we require M to be invertible
- Thus M and M^{-1} both must have all integer entries
- Since $\det(M^{-1}) = (\det(M))^{-1}$ we have if $D = \det(M) \Rightarrow D, D^{-1} \in \mathbb{Z}$ so $D = \pm 1$
- For simplicity we restrict to $D = 1$ which implies F and f are orientation preserving.
- So we assume henceforth that

$$M \in SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

some variance in the meaning of this