DS 19

1A

Recall the set up

- $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and $\det = 1$ yields $L : \mathbb{R}^2 \to \mathbb{R}^2$ via $L(x,y) = M\begin{pmatrix} x \\ y \end{pmatrix}$

- and the 2-torus in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and L descends to $f : \mathbb{T}^2 \to \mathbb{T}^2$

- $$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{L} & \mathbb{R}^2 \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 & \xrightarrow{f} & \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2 \end{array}$$

- Note: For the topology on $\mathbb{T}^2$, U is open $\Leftrightarrow$ $\pi^{-1}(U)$ is open in $\mathbb{R}^2$ or use the metric upstairs to get one downstairs

NOTE for the topologists

$H_1(T^2; \mathbb{R}) = \mathbb{R}^2$

and $f_*: H_1(T^2; \mathbb{R}) \circlearrowleft$ is $f_* = M$ the matrix

and $H_1(T^2; \mathbb{Z}) = \mathbb{Z}^2$

and $f_* = M$ again (note $M: \mathbb{Z}^2 \to \mathbb{Z}^2$)
bijectively

(2)

- We can dynamically classify these matrices using the eigenvalues

- Recall $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T \pm \sqrt{T^2 - 4}}{2}$ since $D = 1$

- CASES (a) $T = 0$ $\lambda = \frac{\pm 2i}{2} = \pm i$ using the real Jordan form $C^{-1}MC = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \Lambda$ note

$$C^{-1}M^4C = \Lambda^4 = I \quad \text{so } M^4 = I$$

So $f_M : \mathbb{T}^2 \to \mathbb{T}^2$ also satisfies $f_M^4 = id$

(This is called finite order, every point is periodic) eg $M = \begin{pmatrix} -2 & -5 \\ 1 & 2 \end{pmatrix}$

(b) $T = 1$, $\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

so $\exists C$ $\quad C^{-1} M C = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \Lambda$

Notice that Λ is rigid rotation by $\pi/3$.

so $\Lambda^6 = Id$ so $M^6 = Id$ also and

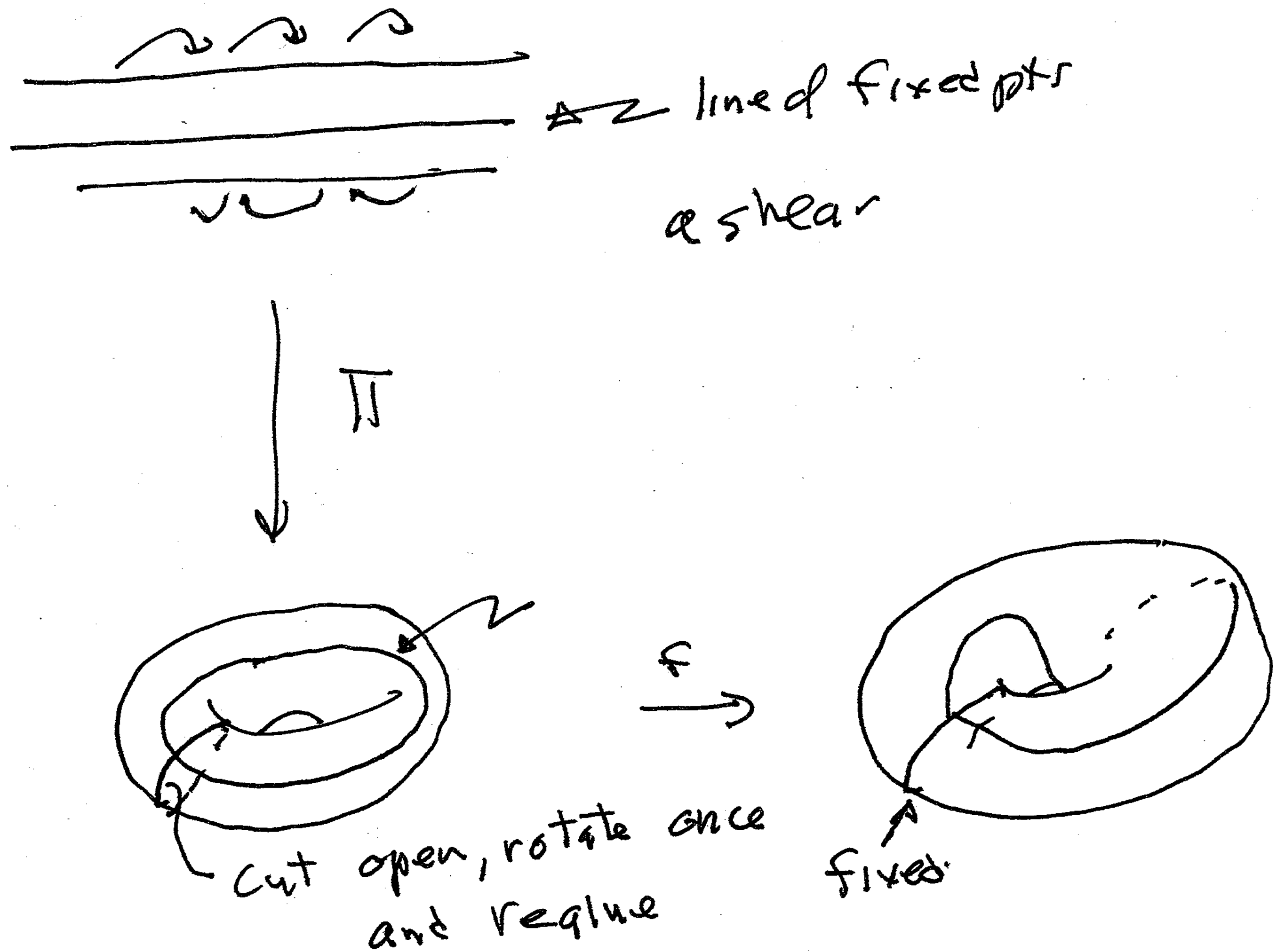
so $f^6 = id$ also, finite order again

(c) $T = 2$: $\lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1$ repeated

$C^{-1} M C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow M = id$

$C^{-1} M C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

So after change of coordinates L looks like



Dehn Twist

(d) $T > 2$ $\qquad \lambda = \dfrac{T \pm \sqrt{T^2 - 4}}{2}$

$\lambda_1 = \dfrac{T + \sqrt{T^2 - 4}}{2} > 1$ since $T > 2$

and an easy computation yields

$0 < \lambda_2 < 1$ $\qquad \Rightarrow \vec{0}$ is saddle point for L (*)

→ other cases: $T < 0$ are similar (HW)

DEF: If $M \in SL_2(\mathbb{Z})$ has $|\mathrm{trace}(M)| > 2$

$\Rightarrow f_M : \mathbb{T}^2 \to \mathbb{T}^2$ is called a hyperbolic toral automorphism.

- We focus on $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, with other hyperbolic toral automorphisms being similar and $f : \mathbb{T}^2 \circlearrowleft$ is its induced map on the two torus

- We first study the eigenvalues and vectors in more detail

- $\lambda = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$, both irrational numbers $\approx 2.618, .3819$

$$\lambda_1 = \frac{3-\sqrt{5}}{2} \qquad \lambda_2 = \frac{3+\sqrt{5}}{2}$$

Picture of Arnol'd's "Cat map" = Thom's toral automorphism.

For the eigenvectors $\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

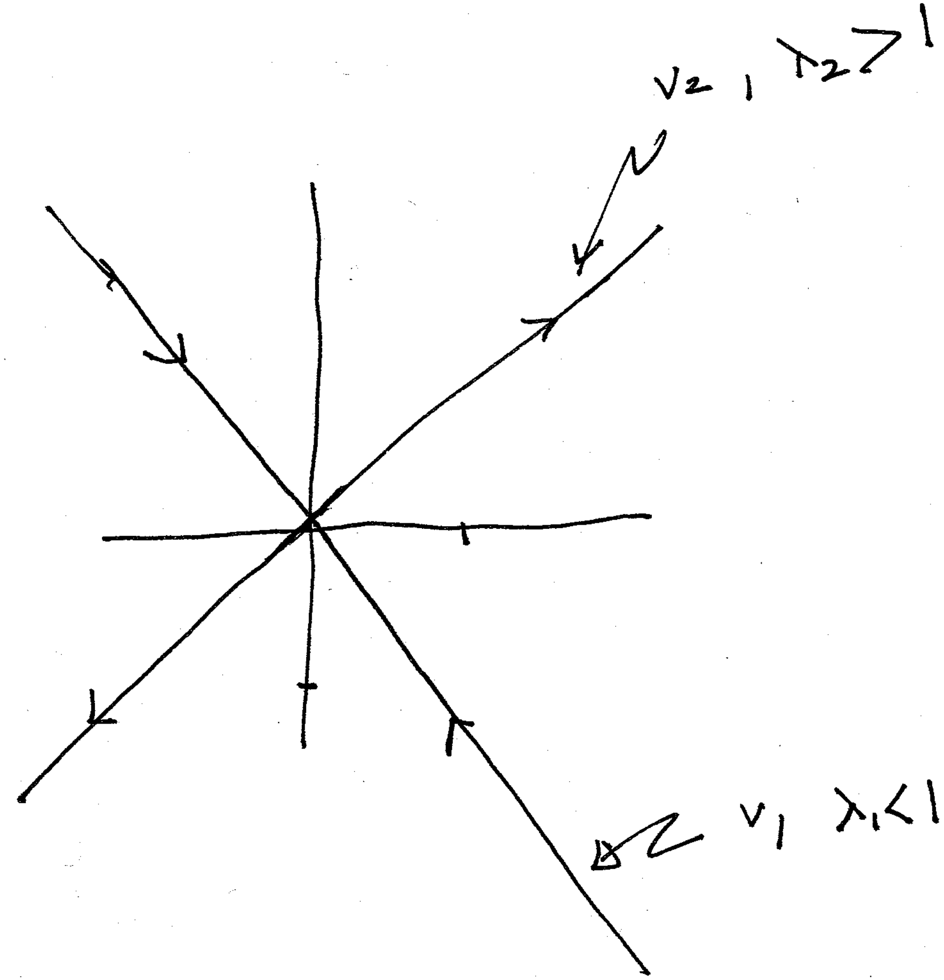
so $(2-\lambda) v_1 + v_2 = 0$

$$\vec{v} = \begin{bmatrix} 1 \\ \lambda - 2 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3-\sqrt{5}}{2} - \frac{4}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix} \approx \begin{bmatrix} 1 \\ -1.6 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3+\sqrt{5}}{2} - \frac{4}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ .61 \end{bmatrix}$$

Notice $\vec{v}_1 \cdot \vec{v}_2 = 0$ so $\vec{v}_1 \perp v_2$, has to happen since M is symetric



A crucial feature is that the slopes of both the eigen vectors are irrational

What are the fixed points of f?

Since $L(\vec{0}) = \vec{0} \Rightarrow f(\vec{0}) = \vec{0}$

if $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is another point in $\mathbb{R}^2$ with $\pi\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

(recall $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ can be write $\pi(x,y) = (x \bmod 1, y \bmod 1)$)

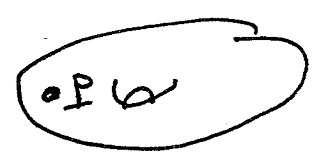
then $M\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix}$ $\quad \begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^2$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} \Rightarrow \begin{matrix} x_0 + y_0 = m \\ y_0 = n \end{matrix} \Rightarrow x_0 = m - n$$

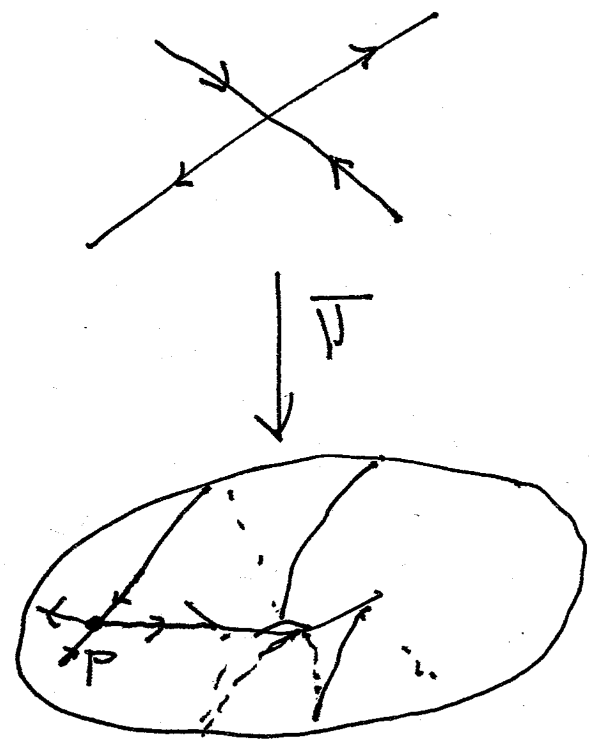
so $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{Z}^2$ so $\pi\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \pi\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix}\right)$

$= \pi\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\underline{\underline{\text{So}}}$ $\pi(\vec{0})$ is unique fixed point of f. Let $P = \pi(\vec{0})$



Recall that $W^s(\vec{0})$ = line through $\vec{v}_1$ and origin
$W^u(\vec{0})$ = " " $\vec{v}_2$ and origin

<u>Lemma</u> $W^s(P) = \pi(W^s(\vec{0}))$ and $W^u(P) = \pi(W^u(\vec{0}))$ and both are dense in $\mathbb{T}^2$.



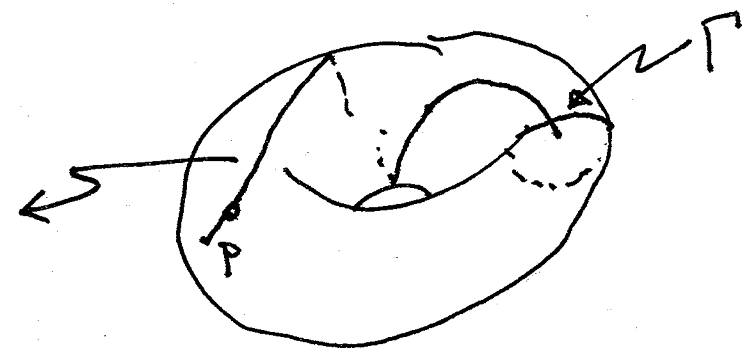
<u>Proof</u>: The equalities follow from the commutivity of

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{L} & \mathbb{R}^2 \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T}^2 & \xrightarrow{f} & \mathbb{T}^2 \end{array}$$

Now for the density, consider $W^u(P)$, the other case is similar

In the torus choose a circle, say $\pi(\{1/2\} \times [0,1])$

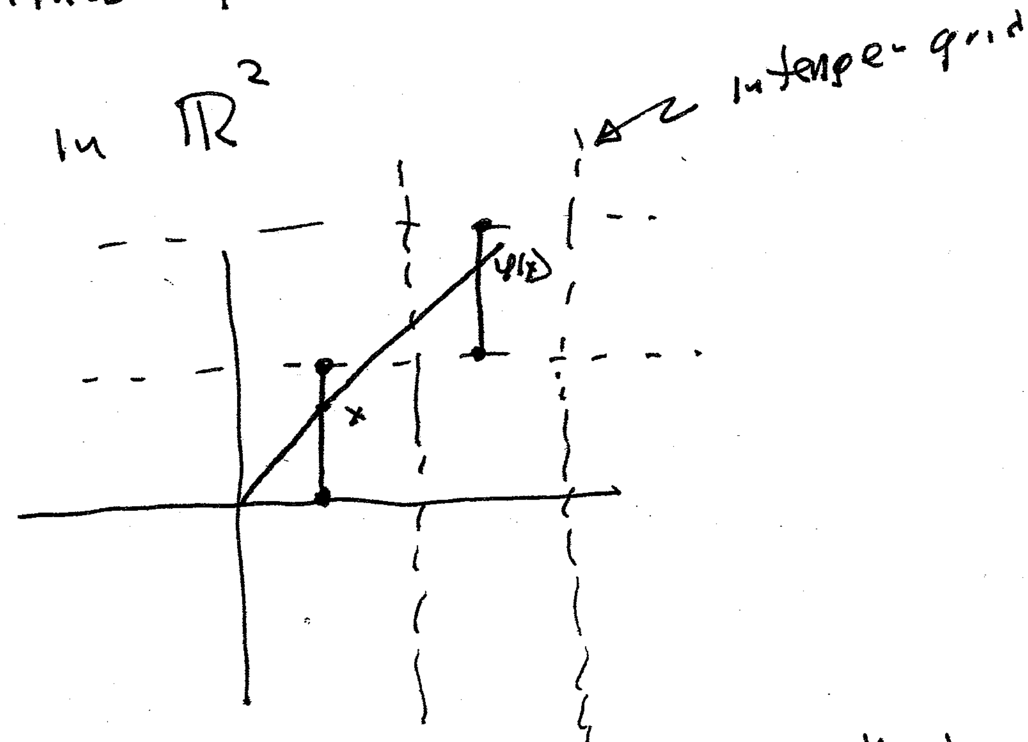
call it Γ



If $W_+^u(P)$ hits Γ in some point x, let $\varphi(x)$ be the next time it hits (we just go out $W^u(P)$ in one direction and call it $W_+^u(P)$)

This defines $\varphi : W_+^u(p) \cap \Gamma \to W_+^u(p) \cap \Gamma$

picture in $\mathbb{R}^2$



The crucial point is that since $W_+^u(p)$ has irrational slope, φ corresponds to an irrational rigid rotation on the circle Γ and so $W_+^u(p) \cap \Gamma$ is dense in Γ which implies $W_+^u(p)$ is dense in $\mathbb{T}^2$ ∎

<u>Theorem</u> Let $f: \mathbb{T}^2 \to \mathbb{T}^2$ be induced by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ on $\mathbb{R}^2$ $\Rightarrow$ f has the following properties

(1) f is transitive

(2) The collection of periodic points of f is dense in $\mathbb{T}^2$

(3) f has sensitive dependence on initial conditions

<u>Proof</u>: next time