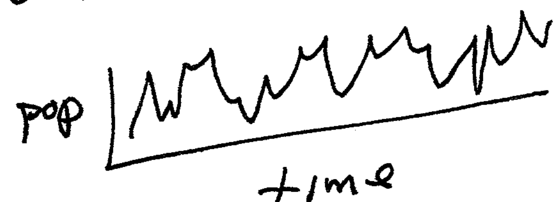
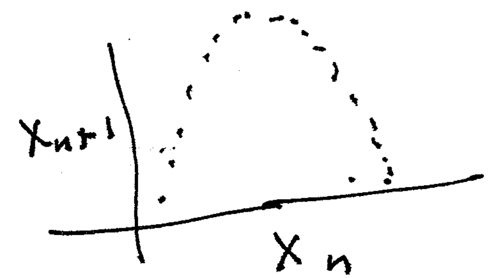
BACK TO POPULATION MODEL

- if the time evolution of a population looks like



does it mean that it is stochastic, random or just a chaotic iterated simple model ($\mu = 4$)?

- BUT if plot x_n vs x_{n+1} see simple structure



- Another big source of dynamical systems is iterative numerical algorithms – Newton's method, iterative linear solvers, ...

MAIN CONCEPTS and ideas

(1) Study asymptotic behaviour

(2) Recurrence - interesting states or points are one's that come back to or near their initial conditions

(3) Indecomposibility - Prime decompositions, find subpieces that can't be dynamically decomposed

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These are formalized in various ways

Reminder: in metric space $x_n \Rightarrow x_0$ implies $f(x_n) \to f(x_0)$ for all seques $\Leftrightarrow$ f is continuous.

3

Assume X is compact, metric and $h: X \to X$ is a homeomorphism.

- $o(x,h) = \{\dots, h^{-2}(x), h^{-1}(x), x, h(x), h^2(x), \dots\}$

- omega-limit set (the future)

  $\omega(x) = \{y : \exists\, n_i \to \infty \text{ with } h^{n_i}(x) \to y\}$

- alpha limit set (the past)

  $\alpha(x) = \{y : \exists\, n_i \to -\infty \text{ with } h^{n_i}(x) \to y\}$

- $Y \subseteq X$ is forward invariant if $h(Z) \subseteq Z$

$h: \mathbb{R}^2 \to \mathbb{R}^2 \quad h(x,y) = (\tfrac{1}{2}x, \tfrac{1}{2}y)$

$Y = \{z : |z| < 1\}$ is fwd. inv.

Lemma $\omega(x)$ is compact, forward invariant set

Pf Since $\omega(x) \subseteq X$ which is compact, it suffices to show $\omega(x)$ is closed to get it is compact

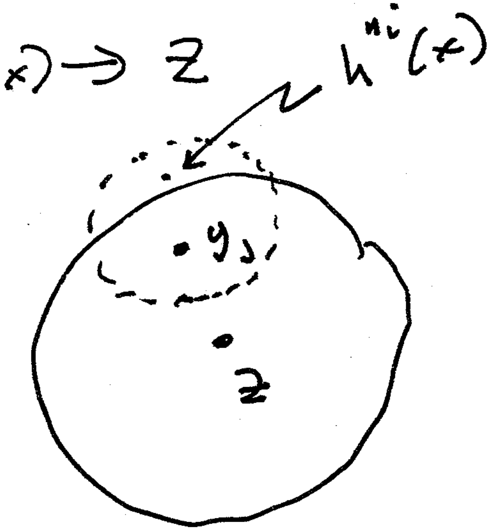
we show $(\omega(x))^c = X - \omega(x)$ is open

Pick $z \in X - \omega(x)$, claim: $\exists j$ so that $B_{\frac{1}{j}}(z) \subseteq X - \omega(x)$. If not, $\forall j$, $\exists y_j \in B_{1/j}(z)$

Now $y_j \in \omega(x)$ so $\exists n_j$ with $d(h^{n_j}(x), y_j) < \frac{1}{j}$

and so $d(h^{n_j}(x), z) \leq d(h^{n_j}(x), y_j) + d(y_j, z)$

$\leq \frac{1}{j} + 1/j = \frac{2}{j}$. Thus $h^{n_j}(x) \to z$



So $z \in \omega(x)$, a contradiction

- Forward ~~in~~ invariant is easier

If $y \in \omega(x) \Rightarrow \exists n_i \to \infty$ $h^{n_i}(x) \to y$

and so by continuity $h^{n_i+1}(x) \to h(y)$

so $h(y) \in \omega(x)$. Thus $h(\omega(x)) \subseteq \omega(x)$ $\blacksquare$

- A set Z is called completely invariant if $h(Z) = Z$ or equivalently

$$h(Z) \subseteq Z$$

and $h^{-1}(Z) \subseteq Z$

- The proof above shows that $\omega(x)$ is completely invariant and an analogous proof applies to $\alpha(x)$

- notice that the proof above shows that $\omega(x)$ is closed whenever X is metric and perhaps not compact.

- If X is not compact though, $\omega(x)$ could be empty: $h: \mathbb{R} \circlearrowleft$ $h(x) = x+1$ 
  $\omega(x) = \emptyset$ for all $x \in \mathbb{R}$.

- But if X is compact $\Rightarrow \omega(x) \neq \emptyset \ \forall x \in X$

proof: $\{x, f(x), f^2(x), \dots\}$ is an infinite set in the compact space X, so there $\exists$ a convergent subsequence i.e. a point y with $f^{n_i}(x) \to y$ as $n_i \to \infty$ so $y \in \omega(x)$.

examples

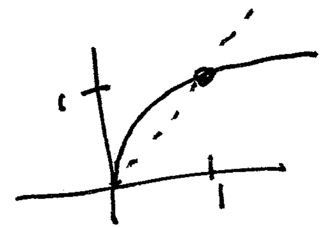
(1) $h(x) = \sqrt{x}$ $\quad h: [0,\infty) \circlearrowleft$ (cheat, not cpt.)

$x<1 \quad h(x) > x$

$x>1 \quad h(x) < x$

$h(0) = 0$

$h^n(x) \to 1$ if $x \neq 0$

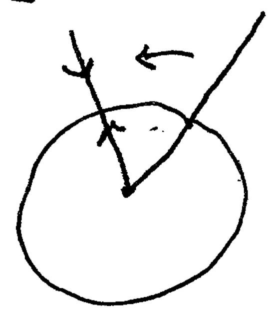


so $x \in (0,\infty)$, $\omega(x) = 1$

(2) In polar coordinates

$h(r,\theta) = (\sqrt{r}, \theta + \omega)$

$0 < \omega < 2\pi$



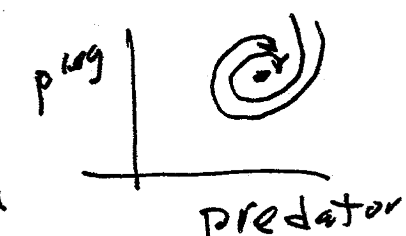
h preserves rays through origin

if $\omega \notin \mathbb{Q}$, $\cancel{z} \in \mathbb{R} - \{0\}$

then $\omega(z) = \{|z| = 1\}$

if $\omega \in \mathbb{Q}$, $z \in \mathbb{R} - \{0\}$,

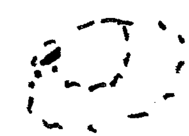
$\omega(z)$ = a periodic orbit on $\{|z| = 1\}$.



(3) population: $\omega(p)$ = asymptotic equilibrium (if it exists)

- $\boxed{\text{Recurrence}}$

- x is forward recurrent if $\exists\, n_i \to \infty$ with $\phi_h^{n_i}(x) \to x$



- equivalently, $x \in \omega(x)$

- Fixed points and periodic points are recurrent

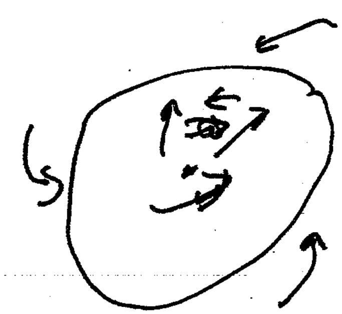
- $R_\omega : S^1 \circlearrowleft,\ \theta \mapsto \theta + \omega$
  - $\omega \in \mathbb{Q}$, every point is periodic $\Rightarrow$ recurrent
  - $\omega \notin \mathbb{Q}$, $\omega(x) = S^1$ every point
    so $x \in \omega(x)$ every pt.

- $h : \mathbb{R}^2 \circlearrowleft \quad h(r, \theta) = (\sqrt{r}, \theta + \omega)$

  (1) $h(\vec{0}) = \vec{0}$ is fixed $\Rightarrow$ recurrent
  (but "unstable", nearby points run away)

  (2) $|z| = 1 \Rightarrow$ recurrent since $h|_{S^1} = R_\omega$

  (3) all other z are not recurrent
  cause $\omega(z) \subseteq S^1$ and $z \notin S^1$



- If x is recurrent $\Rightarrow h(x)$ is recurrent

Proof $\quad h^{n_i}(x) \to x \Rightarrow h^{n_i+1}(x) \to h(x)$

- So collection of Recurrent points is completely invariant - is it compact (HW, tricky)

- As with $\omega(x)$, it is possible there are no recurrent points when X is not cpt $h(x) = x+1$, $h: \mathbb{R} \to \mathbb{R}$

- We will see shortly that when X is compact there are always recurrent points.

Indecomposibility

We have seen the importance of compact invariant sets - in some sense they provide the pieces of a dynamical system.

DEF: Given $h: X \circlearrowleft$ a homeomorphism it is called <u>minimal</u> if it has no proper, non-empty compact invariant subsets.

OR $h(Y) = Y \Rightarrow Y = X$ or $Y = \emptyset$
with Y cpt

Example: • periodic points
- $R_\omega : S^1 \circlearrowleft$ $R_\omega(\theta) = \theta + \omega$, $\omega \notin \mathbb{Q}$
$\Rightarrow (S^1, R_\omega)$ is a minimal set

DEF: If $Z \subseteq X$ and $h(Z) = Z$ then Z is called a minimal set for h if $(Z, h|_Z)$ is minimal example $h(r,\theta) = (\sqrt{r}, \theta + \omega)$ $\omega \notin \mathbb{Q}$ then

$\{|z|=1\}$ is a minimal set for h 

<u>Theorem</u>: If $h : X \circlearrowleft$ is a homeomorphism of a compact metric $X \Rightarrow h$ has a minimal set

<u>Proof</u>: Uses Zorn's Lemma (equivalent to Axiom of choice)

A quick Review:

Say P is a partially ordered set $(P, \leq)$

a descending chain is a linearly ordered subset $X_1 \geq X_2 \geq X_3 \geq \ldots$

if every descending chain has a lower bound i.e. Z with $X_i \geq Z$ $\forall i \Rightarrow$ Zorn's Lemma says P has at least one minimal element i.e. $\exists W$ so that there is no $X \in P$ with $X \leq W$.

Now for the proof:

Let P be the collection of nonempty, compact h-invariant subsets of X ordered by inclusion and let $X_1 \supseteq X_2 \supseteq \dots$ be a descending chain and $Z = \bigcap_{i=1}^{\infty} X_i$. Then since each X_i is compact, a basic topology theorem says Z is compact and non-empty. Since $h(X_i) = X_i$ $\forall i$, it is easy to check that $h(Z) = Z$, Thus $Z \in \mathcal{P}$ and is a lower bound for the chain. So by Zorn's lemma, $\mathcal{P}$ has a minimal element i.e. W that contains no proper, ~~sub~~ nonempty h-invariant compact sets. $\blacksquare$

Next time more on minimality