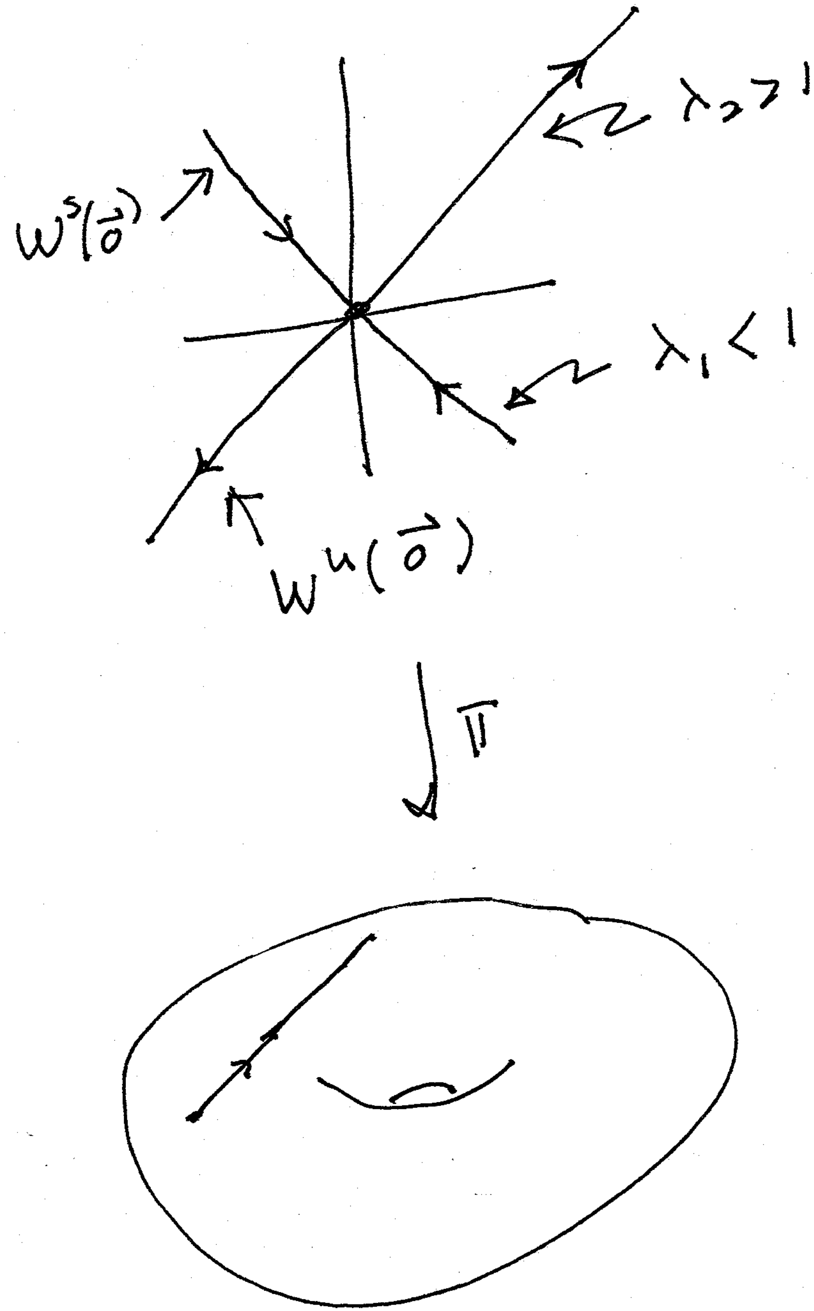
DS20

Recall the set up

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}"] \arrow[d, "\pi"'] & \mathbb{R}^2 \arrow[d, "\pi"] \\ \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \arrow[r, "f"] & \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2 \end{tikzcd}$$

$P = \pi(\vec{0})$ is the unique fixed point of f

Eigenvectors of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$



$W^u(\underline{P}) = \pi(W^u(\vec{0}))$

$W^s(\underline{P}) = \pi(W^s(\vec{0}))$

Each Branch of $W^s(\underline{P})$ and $W^u(\underline{P})$ is dense in $\mathbb{T}^2$

Action in eigen coordinates

for any $\vec{w} \in \mathbb{R}^2$ we may write

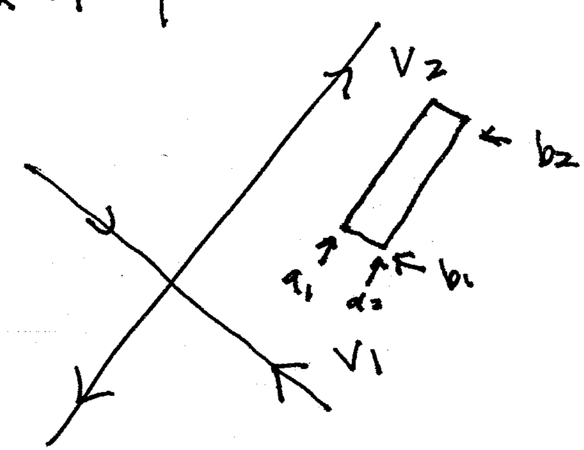
$$\vec{w} = \alpha \vec{v}_1 + \beta \vec{v}_2$$

and then

$$A^n(\vec{w}) = \alpha \lambda_1^n \vec{v}_1 + \beta \lambda_2^n \vec{v}_2$$

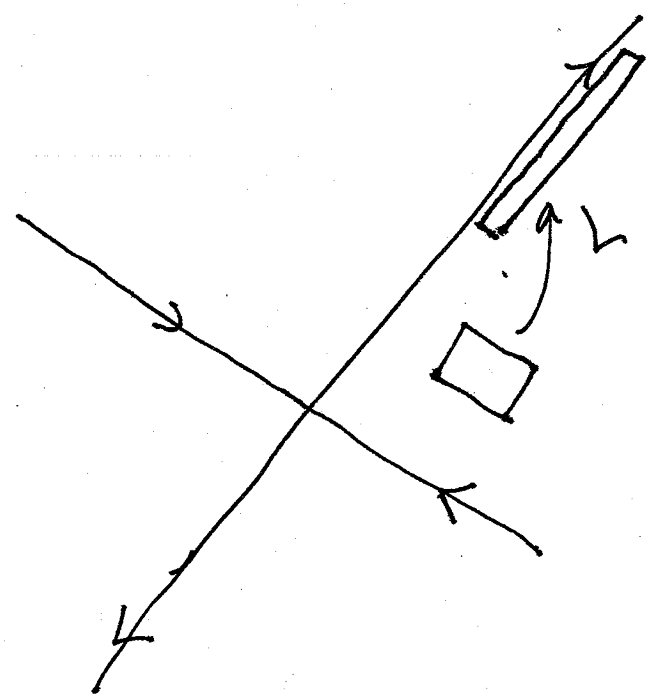
A Box $B(a_1, a_2, b_1, b_2)$

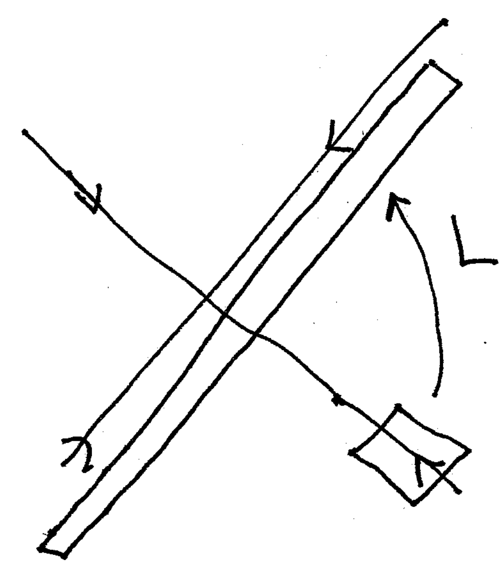
$= \{ \vec{w} = \alpha \vec{v}_1 + \beta \vec{v}_2 : a_1 \le \alpha \le a_2,\ b_1 \le \beta \le b_2 \}$



Fundamental property : Let $L(\vec{x}) = M\vec{x}$

$$L^k(B(a_1, a_2, b_1, b_2))$$
$$= B(\lambda_1^k a_1, \lambda_1^k a_2, \lambda_2^k b_1, \lambda_2^k b_2)$$





<u>Theorem</u>: Let $f: \mathbb{T}^2 \circlearrowleft$ be induced by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on $\mathbb{R}^2$

(1) The collection of periodic orbits of f is dense in $\mathbb{T}^2$

(2) f is transitive

(3) f has sensitive dependence on initial conditions

Proof: (1) For each k, let

$$S_k = \left\{ \left(\frac{\alpha}{k}, \frac{\beta}{k} \right) : 0 \le \alpha, \beta < k \right\}$$

Note that elements of S_k are not in reduced form.

So, for example, $(1/2, 1/3) = \left(\frac{3}{6}, \frac{2}{6}\right) \in S_6$

Now S_k is finite with k^2-elements

and further for $\left(\frac{\alpha}{k}, \beta/k\right) \in S_k$

$$\vec{y} = M\begin{pmatrix}\alpha/k\\ \beta/k\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha/k\\ \beta/k\end{pmatrix} = \begin{pmatrix}(2\alpha+\beta)/k\\ \alpha+\beta/k\end{pmatrix}$$

Thus while ... it is not necessary that $\vec{y} \in S_k$, for some (m,n) $\vec{y} + \begin{pmatrix}m\\ n\end{pmatrix} \in S_k$.

Thus $f(\pi(S_k)) \subseteq \pi(S_k)$

Similary $M^{-1}\begin{pmatrix}\alpha/k\\ \beta/k\end{pmatrix} = \begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}\alpha/k\\ \beta/k\end{pmatrix} = \begin{pmatrix}(\alpha-\beta)/k\\ (-\alpha+2\beta)/k\end{pmatrix}$

and so $f^{-1}(\pi(S_k)) \subseteq \pi(S_k)$

Thus $f(\pi(S_k)) = \pi(S_k)$ and $f|_{\pi(S_k)}$ is a bijection.

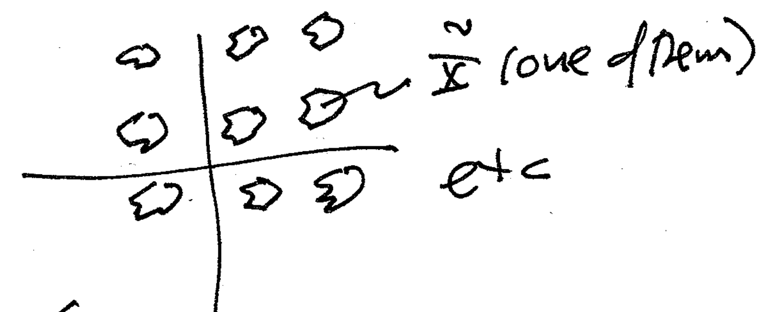
Thus every point in $\pi(S_k)$ is periodic of period $\leq k^2$

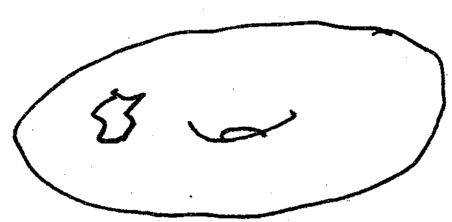
Thus $\pi\left(([0,1) \cap \mathbb{Q})^2\right) \subseteq \bigcup_{k \in \mathbb{N}} \pi(S_k) \subseteq \mathrm{Per}(f)$

and so $\overline{\mathrm{Per}(f)} = \mathbb{T}^2$.

(2) - We need a definition: For a set $X \subseteq \mathbb{T}^2$ with diameter $< 1/2$, a lift is a set $\tilde{X} \subseteq \mathbb{R}^2$ so that $\pi|_{\tilde{X}}$ is a homeomorphism $\tilde{X} \to X$.

- any such X has many lifts all differing by elements of $\mathbb{Z}^2$



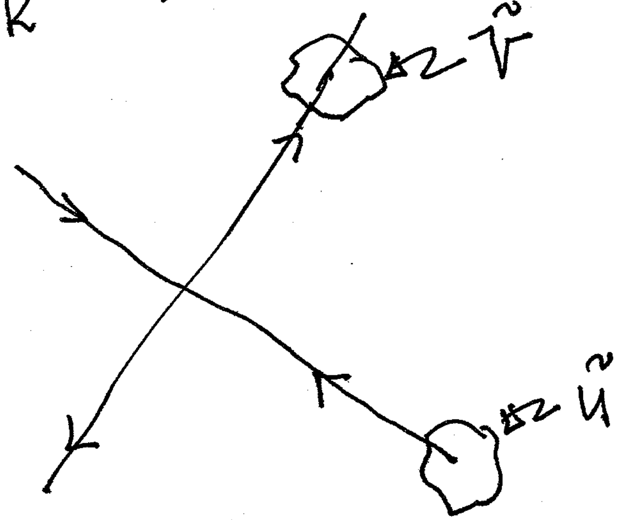


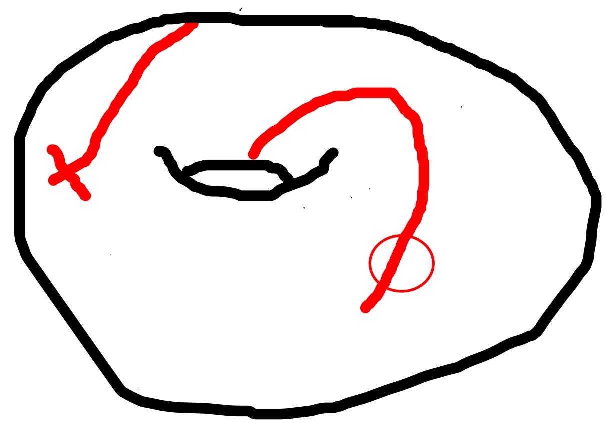
The total lift is the union of all the $\tilde{X}$.

- We show that for any two open sets $\mathcal{U}, \mathcal{V} \subseteq \mathbb{T}^2$ there exists $n \geq 0$ with $f^n(\mathcal{U}) \cap \mathcal{V} \neq \phi$.

- First note that it suffices to consider $\mathcal{U} = B_{\varepsilon_1}(x)$, $\mathcal{V} = B_{\varepsilon_2}(y)$ with $\varepsilon_1, \varepsilon_2 < 1/2$.

- Now each branch of $W^s(P)$ is dense in $\mathbb{T}^2$. This means that $\mathcal{U}$ has a lift $\tilde{\mathcal{U}} \subseteq \mathbb{R}^2$ with $W^s(\vec{0}) \cap \tilde{\mathcal{U}} \neq \phi$. And similarly, $\exists$ lift of $\mathcal{V}$ with $\tilde{\mathcal{V}} \subseteq \mathbb{R}^2$ with $\tilde{\mathcal{V}} \cap W^u(\vec{0}) \neq \phi$.





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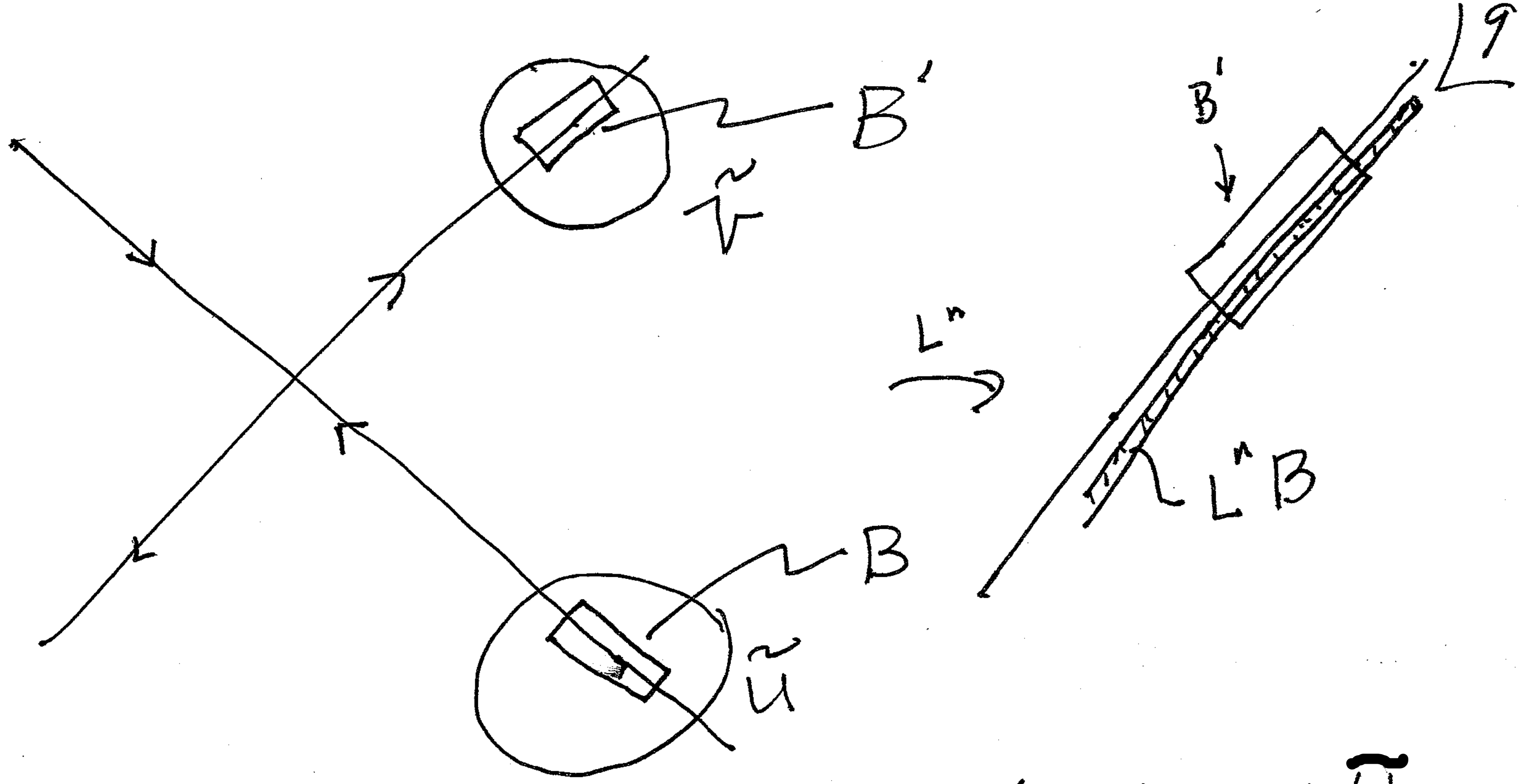


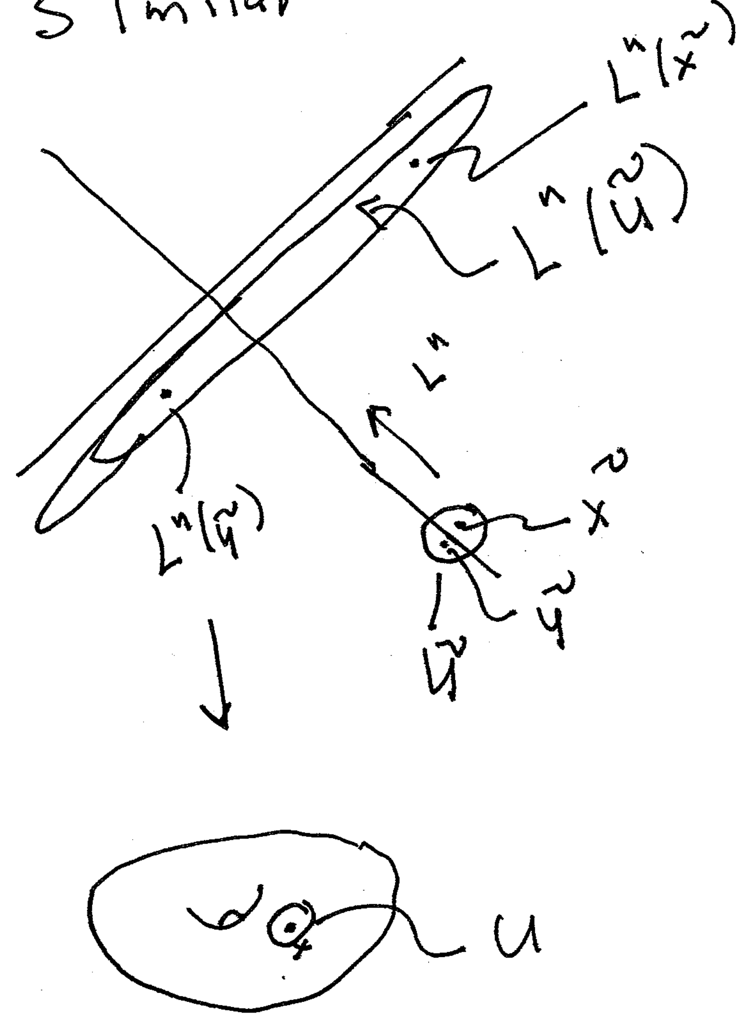
Now Find a_1, a_2 $b_1 < 0 < b_2$ with $B(a_1, a_2, b_1, b_2) \subseteq \tilde{U}$

and $a_1' < 0 < a_2'$, b_1', b_2' with $B'(a_1', a_2', b_1', b_2') \subseteq \tilde{V}$

and n so that $\lambda_2^n b_2 > b_1'$ and $\lambda_1^n a_1 < a_2'$

and Then $L^n(\tilde{U}) \cap \tilde{V} \neq \phi$ and so $f^n(U) \cap V \neq \phi$.

(3) Sensitive Dependence on initial conditions is similar



and leave for HW

These 3 properties: Dense periodic pts, transitive and SDIC are all shared by subshifts of finite types. So it should be no suprise that there exists a ssft Σ_A and a continuous, onto $\alpha: \Sigma_A \to \mathbb{T}^2$ with

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \alpha \downarrow & & \downarrow \alpha \\ \mathbb{T}^2 & \xrightarrow{f} & \mathbb{T}^2 \end{array}$$

with α finite to one and one-to-one on a big set.

- The construction is complicated. One constructs Markov Rectangles which serve as an Address System (see figures)

- Similar theorems hold when $M \in SL_n(\mathbb{Z})$

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{M} & \mathbb{R}^n \\ \pi \downarrow & & \downarrow \\ \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n & \xrightarrow{f} & \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n \end{array}$$

and $\operatorname{spec}(M) \cap S^1 = \phi$ so M is hyperbolic (note, since $\det(M) = 1 = \prod_{i=1}^{n} \lambda_i$, some e.values are inside and some outside the unit circle.

from Brin and Stuck

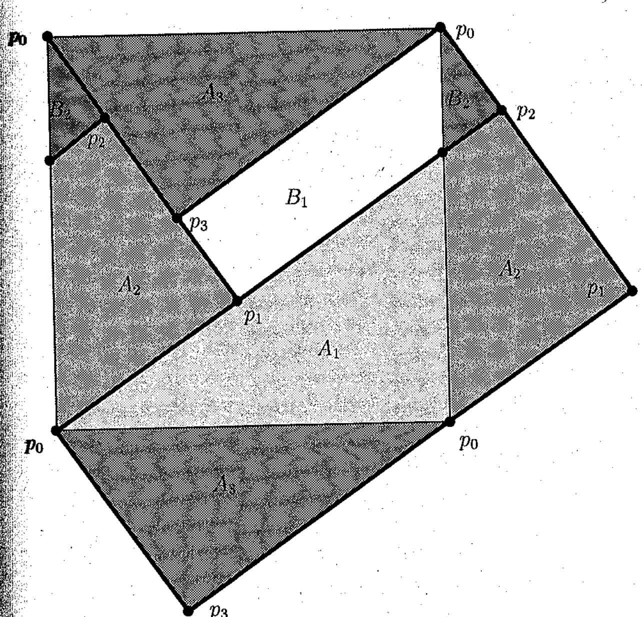


Figure 5.5. Markov partition for the toral automorphism f_M.

product structure "commutator" $[x, y]$, i.e., if $x, y \in R_i$, then $[x, y] \in R_i$. For $x \in R_i$ let $W^s(x, R_i) = \bigcup_{y \in R_i} [x, y]$ and $W^u(x, R_i) = \bigcup_{y \in R_i} [y, x]$. The last condition means that if $x \in \text{int } R_i$ and $f(x) \in \text{int } R_j$, then $W^u(f(x), R_j) \subset f(W^u(x, R_i))$ and $W^s(x, R_i) \subset f^{-1}(W^s(f(x), R_j))$.

The partition of the unit interval $[0, 1]$ into m intervals $[k/m, (k+1)/m)$ is a Markov partition for the expanding endomorphism E_m. The target subshift in this case is the full shift on m symbols.

We now describe a Markov partition for the hyperbolic toral automorphism $f = f_M$ given by the matrix

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

which was constructed by R. Adler and B. Weiss [AW67]. The eigenvalues are $(3 \pm \sqrt{5})/2$. We begin by partitioning the unit square representing the torus $\mathbb{T}^2$ in Figure 5.5 into two *rectangles*: A, consisting of three parts A_1, A_2, A_3; and B, consisting of two parts B_1, B_2. The longer sides of the rectangles are parallel to the eigendirection of the larger eigenvalue $(3 + \sqrt{5})/2$, and the shorter sides are parallel to the eigendirection of the smaller eigenvalue $(3 - \sqrt{5})/2$. In Figure 5.5, the identified points and regions are marked by the same symbols. The images of A and B are shown in Figure 5.6. We subdivide

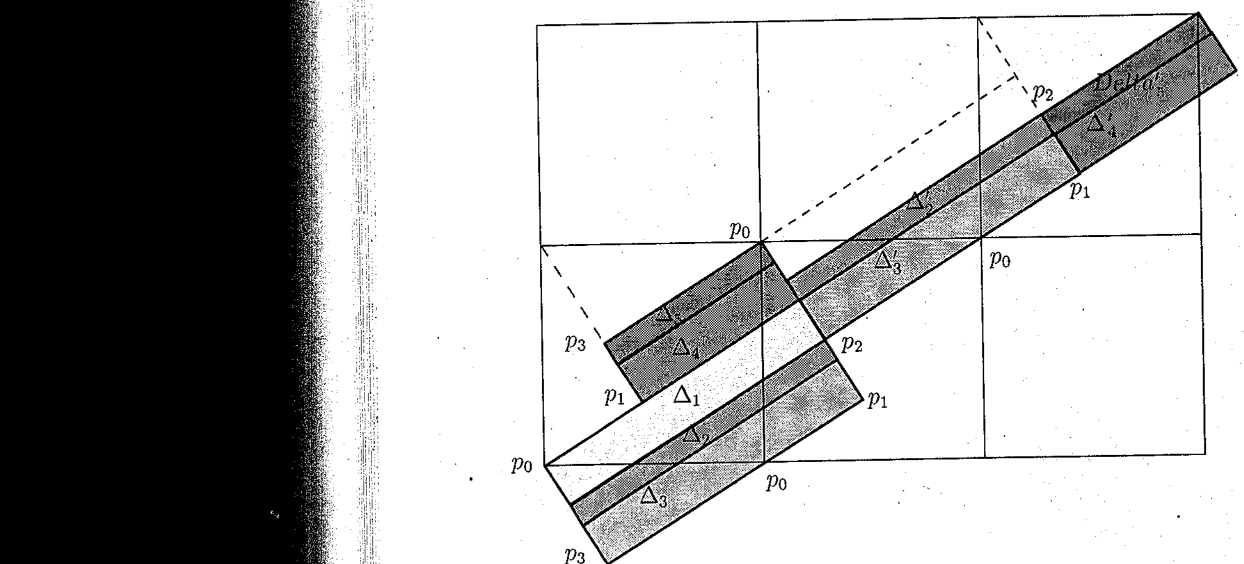


Figure 5.6. The image of the Markov partition under f_M.

A and B into five subrectangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ that are the connected components of the intersections of A and B with $f(A)$ and $f(B)$. The image of A consists of Δ_1, Δ'_3 and Δ'_4; the image of B consists of Δ'_2 and Δ'_5. The part of the boundary of the Δ_i's that is parallel to the eigendirection of the larger eigenvalue is called *stable*; the part that is parallel to the eigendirection of the smaller eigenvalue is called *unstable*. By construction, the partition Δ of $\mathbb{T}^2$ into five rectangles Δ_i has the property that the image of the stable boundary is contained in the stable boundary, and the preimage of the unstable boundary is contained in the unstable boundary (Exercise 5.12.1). In other words, for each i, j, the intersection $\Delta_{ij} = \Delta_i \cap f(\Delta_j)$ consists of one or two rectangles that stretch "all the way" through Δ_i, and the stable boundary of Δ_{ij} is contained in the stable boundary of Δ_i; similarly, the intersection $\Delta_{ij}^{-1} = \Delta_i \cap f^{-1}(\Delta_j)$ consists of one or two rectangles that stretch "all the way" through Δ_i, and the unstable boundary of Δ_{ij}^{-1} is contained in the unstable boundary of Δ_i. Let $a_{ij} = 1$ if the interior of $f(\Delta_i) \cap \Delta_j$ is not empty, and $a_{ij} = 0$ otherwise, $i, j = 1, \ldots, 5$. This defines the adjacency matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$