Flows and Differential equations

A flow is a continuous map $\varphi : X \times \mathbb{R} \to X$ such that for t (writing $\varphi_t(x) = \varphi(t, x)$)

(a) φ_t is a homeomorphism of X for all t.

(b) $\varphi_0 = id$

(c) $\varphi_{s+t}(x) = \varphi_s(\varphi_t(x))$

or $\varphi_{s+t} = \varphi_s \circ \varphi_t$

Note: some books write $\varphi^t(x)$, or $\varphi(t, x)$ or $\varphi(x, t)$ ..

or φ corresponds to a continuous group <u>homomorphism</u>

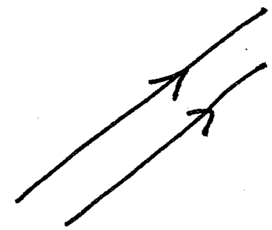
algebraic

$$\mathbb{R} \to \mathrm{Homeo}(X)$$
$$t \mapsto \varphi_t$$

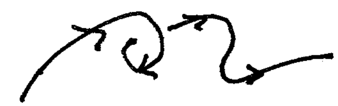
<u>Example</u> on $\mathbb{R}^2$ Pick $\alpha_1, \alpha_2 \in [0,1]$

$$\varphi_t(x_1, x_2) = (x_1, x_2) + t(\alpha_1, \alpha_2)$$

or $\varphi_t(\vec{x}) = \vec{x} + t\vec{\alpha}$



The full orbit of x under φ_t is (or trajectory or flow line)



$$\varphi_{\mathbb{R}}(x) = \{\varphi_t(x) : t \in \mathbb{R}\}$$

forward orbit $\varphi_{\mathbb{R}^+}(x) = \{\varphi_t(x) : t \in \mathbb{R}^+ = [0,\infty)\}$

backwards orbit $\varphi_{\mathbb{R}^-}(x) = \{\varphi_t(x) : t \in \mathbb{R}^- = [-\infty, 0]\}$

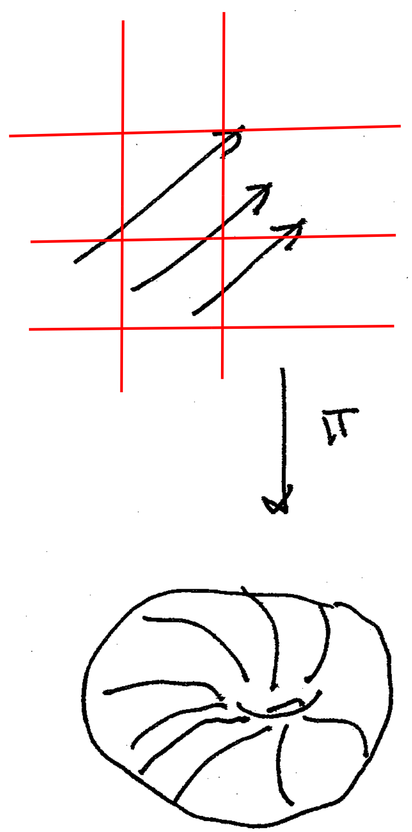
Example on $\mathbb{T}^2$: From befo.

$$\varphi_t(\vec{x}) = \vec{x} + t\vec{\alpha}$$

notice $\varphi_t\left(\vec{x} + \begin{pmatrix} m \\ n \end{pmatrix}\right) = \vec{x} + \begin{pmatrix} m \\ n \end{pmatrix} + t\vec{\alpha} = \varphi_t(\vec{x}) + \begin{pmatrix} m \\ n \end{pmatrix}$

So φ_t descends to $\psi_t : \mathbb{T}^2 \to \mathbb{T}^2$

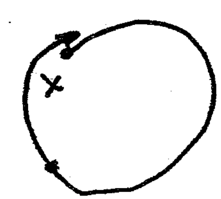
$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\varphi_t} & \mathbb{R}^2 \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T}^2 & \xrightarrow{\psi_t} & \mathbb{T}^2 \end{array}$$



The basic structures and properties from discrete dynamics occur in this setting with a new kinda trivial feature.

- p is a rest point, singularity, zero, fixed point.. if $\varphi_t(p) = p, \ \forall t \in \mathbb{R}$
- ~~x~~ is forward recurrent if $\exists n_i \to \infty$ with $\varphi_{n_i}(x) \to x$

The n_i do *not* have to be integers!



in 3D

- backwards recurrent
- The ω limit set of x is $\omega(x, \varphi_t) = \{y :$
- $\exists n_i \to \infty$ with $\varphi_{n_i}(x) \to y$.
- x is a periodic orbit if $\exists$ $T > 0$ so that $\varphi_T(p) = p$ and x is not a rest point

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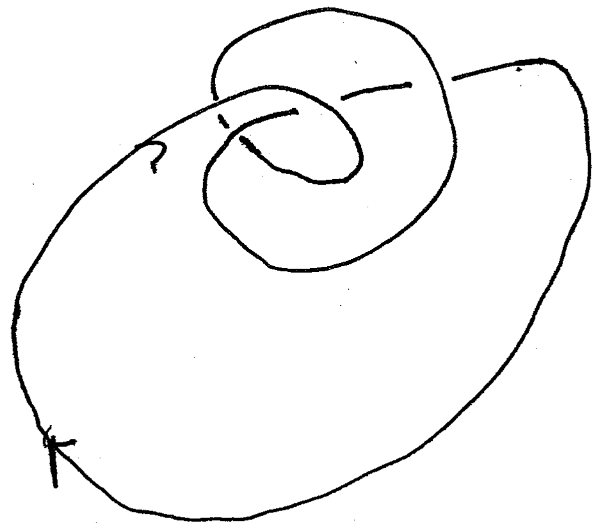
In $\mathbb{R}^3$



- $\overline{X}$ is <u>completly invariant</u> if $\forall x \in \overline{X}$, $\varphi_{\mathbb{R}}(x) \subseteq \overline{X}$ or $\varphi_{\mathbb{R}}(\overline{X}) = \overline{X}$

  or $\varphi_t(\overline{X}) = \overline{X}$, $\forall t$

- $\overline{X}$ is <u>minimal</u> for φ_t if it is completely invariant and for every $x \in \overline{X}$, $\overline{\varphi_{\mathbb{R}}(x)} = \overline{X}$

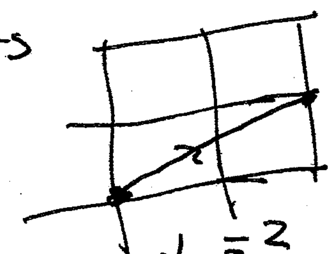
- $\overline{X}$ is <u>transitive</u> if $\exists x$ with $\overline{\varphi_{\mathbb{R}}(x)} = \overline{X}$

- Example on $\mathbb{T}^2$

$$\varphi_t(\vec{x}) = \vec{x} + t\vec{\alpha} \quad \text{on } \mathbb{R}^2$$

descends to ψ_t on $\mathbb{T}^2$

if $\frac{\alpha_1}{\alpha_2} \in \mathbb{Q} \Rightarrow \mathbb{T}^2$ is filled with periodic orbits



$\alpha_1 = 2$
$\alpha_2 = 1$

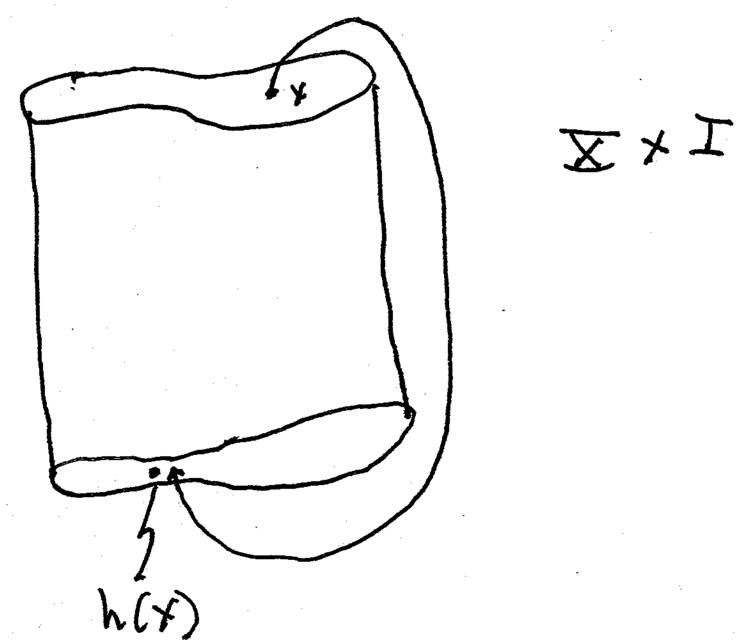
if $\frac{\alpha_1}{\alpha_2} \notin \mathbb{Q} \Rightarrow \mathbb{T}^2$ is minimal under ψ_t

We will explore these notions in more detail but we first study the relations of flows with discrete dynamics and with Differential Equations.

The suspension flow: from discrete to continuous dynamics

Let $h : X \to X$ be a homeomorphism.

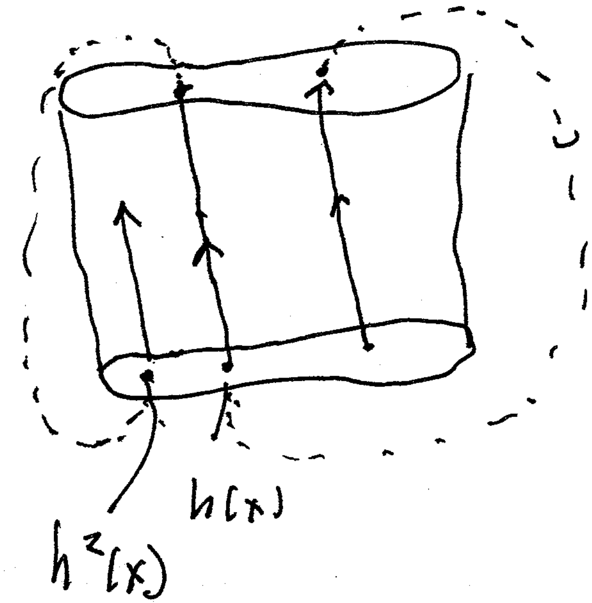
Its <u>mapping torus</u> or <u>suspension manifold</u> is built from $X \times [0,1]$ by identifying $X \times \{0\}$ with $X \times \{1\}$ by h.



So $X_h = X \times [0,1]/\sim$ where

$$(h^{-1}(x), 0) \sim (x, 1)$$

Then suspension flow φ_t on X_h is the projection of the vertical flow on $X \times [0,1]$ to X_h

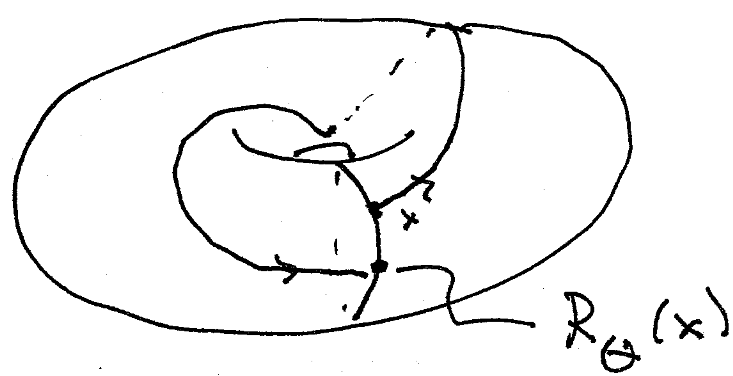


In formulas, $\varphi_t(x) = (h^n(x), s)$ where $n = \lfloor t \rfloor$ (floor)
In forward time $s = \{t\}$ (fractional part)

Example: $X = S^1 \quad h = R_\theta$

the X_h is a torus

and the suspension flow looks like one of the φ_t from above



The suspension flow φ_t shares most of the dynamical properties of h

<u>Theorem</u> h is transitive (minimal) (has dense periodic points) (has SDIC) $\Leftrightarrow$ φ_t does.

Recall that $h : X \to X$, $g : Y \to Y$ are topologically conjugate if $\exists$ a homeomorphism $\alpha : X \to Y$ with

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ \alpha \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{g} & Y \end{array}$$

or

$$\alpha h = g \alpha$$

- Two flows (X, φ_t), (Y, ψ_t) are <u>strongly topologically conjugate</u> if $\exists$ a homeo $\alpha : X \to Y$ with $\alpha \varphi_t = \psi_t \alpha$ for all t

- So for every t, φ_t and ψ_t are topologically conjugate

- This is too strong a requirements
  if we reparameterize a flow, say, making it slower it still has the same dynamics

- So the usual def of top. conj. for flows allows for reparameterization

- Two flows (X, φ_t) and (Y, ψ_t) are topologically conjugate if $\exists$ a homeomorphism $\alpha : X \to Y$ that preserves flow lines and their orientation but not necessarily the parameterization

- So (X, φ_t) is strongly top conj. to a reparameterization of ψ_t, ~~namely~~
$$\alpha \psi_t \alpha^{-1}$$

- <u>Theorem</u>: $h : X \to X$ and $g : Y \to Y$ are topologically conjugate $\Rightarrow$ (X_h, φ_t) and (Y_g, ψ_t) (the suspension flows) are top. conj. (in particular, their mapping toruses are homeomorphic)

<u>**Proof:**</u> Recall the set up

$$\mathbb{X}_h = \mathbb{X} \times [0,1] / (h(x),0) \sim (x,1)$$

$$\mathcal{Y}_g = \mathcal{Y} \times [0,1] / (g(y),0) \sim (y,1)$$

and $\alpha : \mathbb{X} \to \mathcal{Y}$, $\quad \alpha h = g \alpha$

Define $\hat{\alpha} : \mathbb{X} \times [0,1] \to \mathcal{Y} \times [0,1]$
by $\hat{\alpha}(x,t) = (\alpha(x), t)$

We claim $\hat{\alpha}$ induces $\hat{\alpha} : \mathbb{X}_h \to \mathcal{Y}_g$

we need to check it respects the equivalence relations: in $\mathbb{X}_h$ $(h(x),0) \sim (x,1)$

$$\hat{\alpha}(h(x),0) = (\alpha h(x),0) = (g\alpha(x),0)$$

$$\hat{\alpha}(x,1) = (\alpha(x),1)$$

So $\hat{\alpha}(h(x),0) \sim \hat{\alpha}(x,1)$ in $\mathcal{Y}_g$.

Now similarly

$$\hat{\alpha}^{-1} : Y \times [0,1] \to X \times [0,1]$$

via $\hat{\alpha}^{-1}(y,t) = (\alpha^{-1}(y), t)$

induces $\hat{\alpha}^{-1} : Y_g \to X_h$

which is the inverse of $\hat{\alpha}$. So $\hat{\alpha}$ is bijective.

Bicontinuity follows by checking at the indentification points $\boxtimes$

· The converse is false. Nonconjugate homeomorphisms can have topologically conjugate suspension flows. Such homeomorphisms are called <u>flow equivalent</u> (example next lecture)