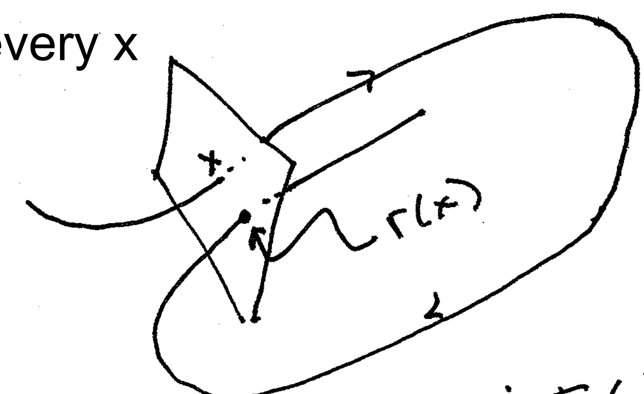
(1)

- From continuous dynamics to discrete dynamics

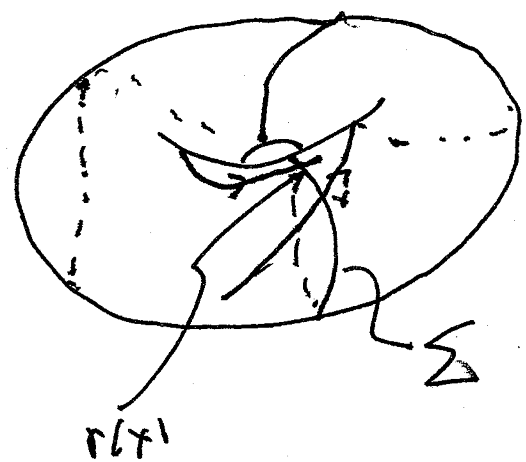
- Let $\varphi_t : X \circlearrowleft$ be a flow and $\Sigma \subseteq X$ be such that

$S(x) = \{ t \in (0, \infty) : \varphi_t(x) \in \Sigma \}$ is a nonempty, unbounded set without limit points for every x



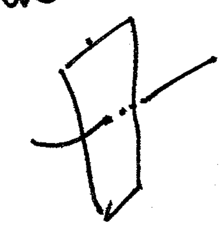
For each $x \in \Sigma$ let $t(x) = \min S(x)$ and $r : \Sigma \to \Sigma$ is $r(x) = \varphi_{t(x)}(x)$, called the <u>first return</u> or <u>Poincare map</u>.

Example: $\varphi_t : \mathbb{T}^2 \mathbb{R}$ is the linear flow example from above. Σ is a circle as shown



- FACT (needs some more advanced topology)

If Σ is a smooth submanifold with φ_t passing transversally through Σ



$\Rightarrow t(x)$ is a continuous function of x and $r : \Sigma \to \Sigma$ is continuous.

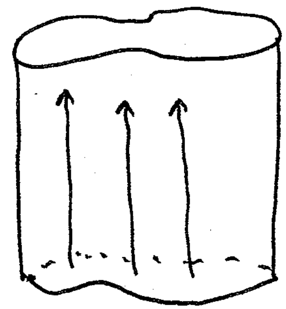
considering the backward flow also we get that
r is a homeomorphism

· Cross sections and suspensions are inverses of each other in the following sense:

(a) If $h: X \to X$ has suspension (X_h, φ_t) then X_h has a cross section Σ and the return map $r: \Sigma \to \Sigma$ is topologically conjugate to h.

(b) If (X, φ_t) is a flow with cross sect Σ with return map r then the suspension built from (Σ, r), namely (Σ_r, ψ_t) is topologically conjugate to (X, φ_t)

Proof (9)



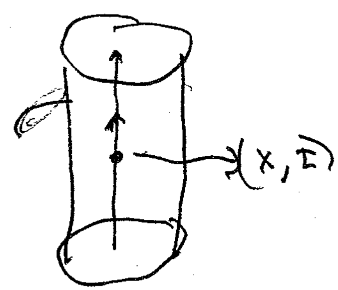
$$X_h = X \times [0,1]/\sim$$

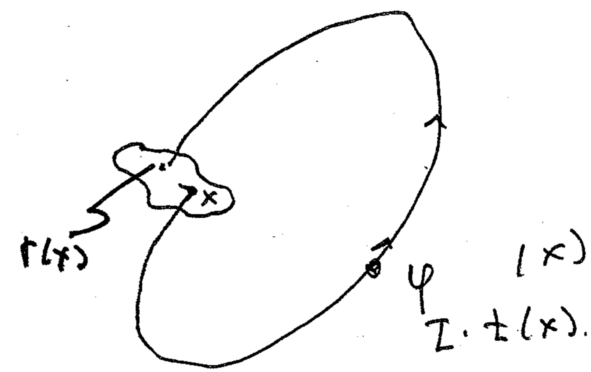
let Σ = the equiv class of $X \times \{0\}$ = the equiv class of $X \times \{1\}$

then Σ is a cross section of the suspension flow with r exactly h.

(b) define $\alpha : \Sigma \times [0,1] \to \overline{\underline{X}}$

via $\alpha(x, I) = \varphi_{I \cdot t(x)}(x)$





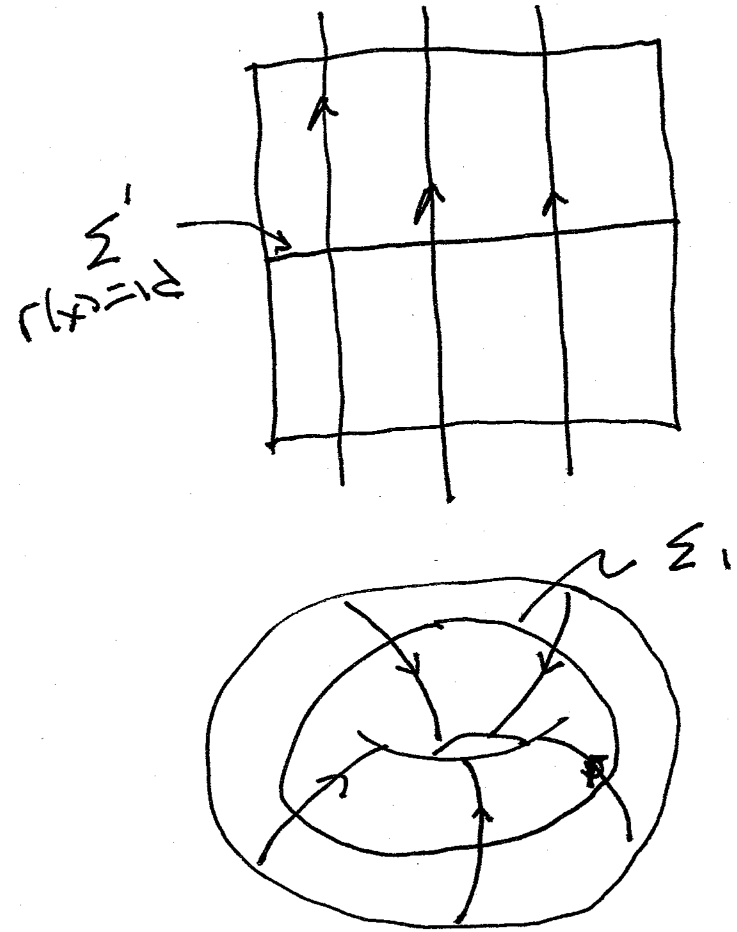
where recall $t(x)$ is the return time of x to Σ.

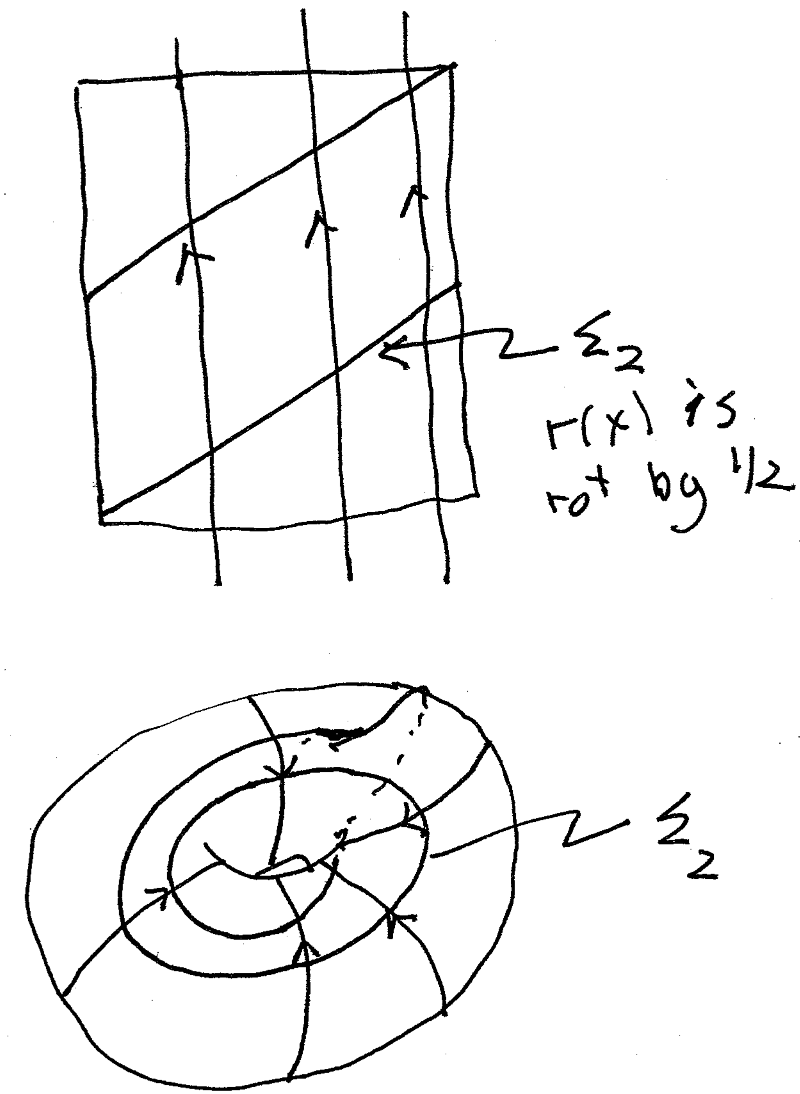
Since $\varphi_{t(x)} = r(x)$ α descends to a map

$\hat{\alpha} : \Sigma \times [0,1]/\sim \;\to \overline{\underline{X}}$ which preserves

flow lines and is a homeomorphism.

an example on $\mathbb{T}^2$





So

- A flow can have different cross sections on which the return maps are non-conjugate (the cross sections are in different cohomology classes)

- Two nonconjugate maps can have suspension flows that are topologically conjugate

Flows and Differential Equations

Given a differential equation with initial values

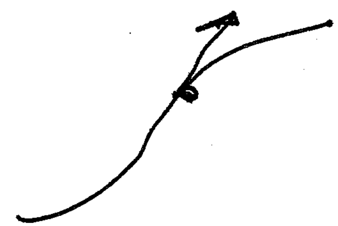
$$\frac{dx}{dt} = \mathbf{F}(x) \qquad x(0) = x_0 \qquad (1)$$

The solution is $x(t; x_0)$ where $x(0; x_0) = x_0$

and $\dfrac{d\,(x(t; x_0))}{dt} = \mathbf{F}(x(t; x_0))$

- NOTE: This familiar notation is a bit confusing. Is x the name of the variable or the function.

Assume now the flow $\varphi: X \times \mathbb{R} \to X$ is differentiable
(so, X is a smooth manifold or $X = \mathbb{R}^n$ or subset thereof)



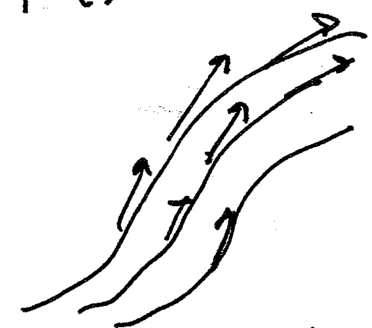
Let $\vec{F}(x) = \left.\frac{\partial \varphi(x,t)}{\partial t}\right|_{t=0}$

FACT: $x(t; x_0) = \varphi_t(x_0)$ solves the DE (1)

Proof $\vec{F}(\varphi_{t_0}(x)) \underset{\substack{\uparrow \\ \text{rename} \\ \text{variables}}}{=} \left.\frac{\partial \varphi_s(\varphi_{t_0}(x))}{\partial s}\right|_{s=0}$

$$= \left.\frac{\partial \varphi_{s+t_0}(x)}{\partial s}\right|_{s=0} \underset{\substack{\uparrow \\ t = s+t_0 \\ s=0 \Rightarrow t=t_0}}{=} \left.\frac{\partial \varphi_t(x)}{\partial t}\right|_{t=t_0}.$$

<u>What about the other way</u>! Given $\vec{F}(x)$ a vector field (say on $U \subseteq \mathbb{R}^n$ an open set or on all of $\mathbb{R}^n$ or on a manifold) is there a flow $\varphi_t(x)$ with $\left.\dfrac{\partial \varphi_t(x)}{\partial t}\right|_{t=0} = \vec{F}(\vec{x})$?



- The answer requires conditions on $\vec{F}$ and uses the basic existence and uniqueness result for DE

Theorem: Assume that $\vec{F}: U \to \mathbb{R}^n$ is a Lipschitz vector field and $x_0 \in U \Rightarrow$ (1) there exists a unique soln $x(t; x_0)$ to $\frac{dx}{dt} = \vec{F}(x)$ defined in some $t \in (-a, a)$ with $a > 0$

(2) The soln $x(t; x_0)$ is continuous as a function of x_0

(3) $x(t; x_0)$ satisfies the group property in its domain of definition

$$x(t+s; x_0) = x(t; x(s; x_0))$$

Just a note on the proof of (1)

$x(t)$ is a soln $\Leftrightarrow$ $x(t) = x_0 + \int^t \frac{dx}{ds}\, ds$

$= x_0 + \int^t \vec{F}(x(s))\,ds$: If $\gamma : [-a,a] \to U$ is Lip (a needs to be specif

Let $\mathcal{F}(\gamma) = x_0 + \int^t \vec{F}(\gamma(s))\,ds$

$\Rightarrow \|\mathcal{F}(\gamma_1) - \mathcal{F}(\gamma_2)\| = \left| \int_0^t \vec{F}(\gamma_1(s)) - \vec{F}(\gamma_2(s))\, ds \right|$

$\leq \int_0^t |\vec{F}(\gamma_1(s)) - \vec{F}(\gamma_2(s))|\, ds$

$\leq a L \|\gamma_1 - \gamma_2\|$

Lip constant of $\vec{F}$

unique

So get contraction and thus fixed point if $aL < 1$

for (2) see DE book for (3)

$$\text{Define } y(\tau) = \begin{cases} x(\tau; x_0) & 0 \le \tau \le s \\ x(\tau - s; x(s, x_0)) & s \le \tau \le s+t \end{cases}$$

$\Rightarrow$ $y(t)$ is a soln (easy to check)

and so $y(\tau) = x(\tau; x_0)$ on $s \le \tau \le s+t$

So $x(s+t; x_0) = x(t; x(s; x_0))$, as required $\blacksquare$

setting $s+t = \tau$

So at least locally in time we get a flow.

Next time standard counter examples and conditions to get global flow