Flows and DE

<u>Thus far</u>: Given open $U \subseteq \mathbb{R}^n$, Lipschitz $\vec{F}$

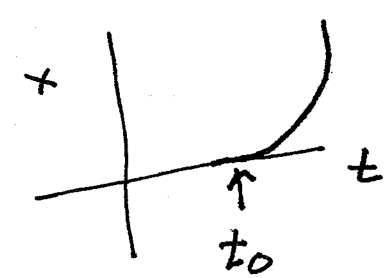
$\Rightarrow \exists$ local flow $\varphi_t(x)$ solving

$\mathring{x} = \vec{F}(x)$ (or integrating $\vec{F}$)

<u>Two examples</u>

(1) Loss of uniqueness when $\vec{F}$ is not Lipschitz

$$\frac{dx}{dt} = 3x^{2/3} \qquad x(t_0) = 0$$



Two soln $\quad x(t) = (t - t_0)^3$

$\qquad x(t) \equiv 0$

(2) Flow only defined on short interval

$$\frac{dx}{dt} = x^2 = f(x)$$

$x(1) = x_0$

Soln is $x(t) = \dfrac{x_0}{1-(t-1)x_0}$

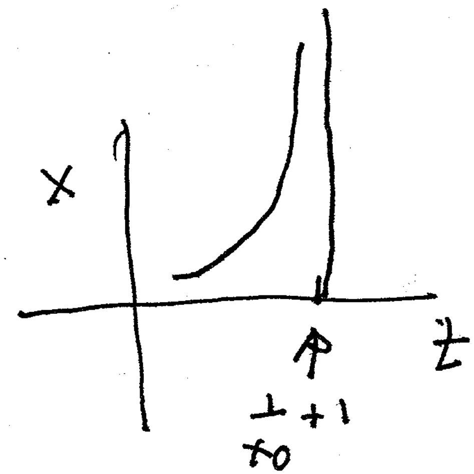
Blows up when

$1-(t-1)x_0 = 0$

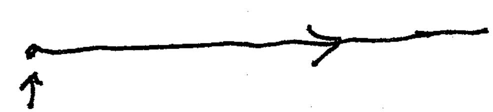
or $t = \dfrac{1}{x_0} + 1$

f is Lip on $(0,k)$
any $k > 0$

$f'(x) = 2x$
$|f'(x)| = 2x \leq 2k$
By MVT
$|f(x_1) - f(x_2)| \leq 2k|x_1 - x_2|$



but in $\mathbb{R}$, $x(t)$



goes to ∞ in finite time.

- This turns out to characterize the main issue when $\varphi_t(x)$ is only local in time

Thm: If $\vec{F}$ is Lipschitz on $\mathbb{R}^n$ and $\exists\, M$ with $|\vec{F}(x)| < M \Rightarrow$ a flow global in time exists which solves $\frac{d\vec{x}}{dt} = \vec{F}(x)$

<u>NOTE</u> $M > |\vec{F}(x)| = \left| \frac{\partial \phi_t(x)}{\partial t} \right|$

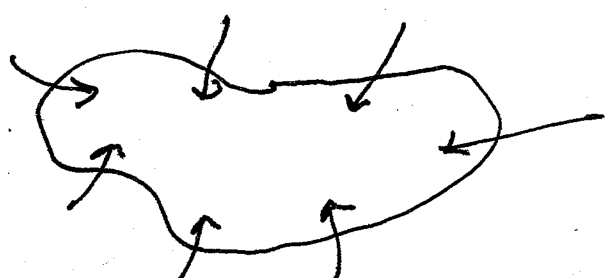
$$\Rightarrow \quad |\varphi_t(x) - x| < M|t|$$

so no infinity in finite time

- This is not a very useful theorem since even Linear DE $\frac{dx}{dt} = Ax$ do not have bounded Right hand sides

- one alternative is to work on compact manifolds and then one always gets a global flow from a C^1- vector field

- In practise in $\mathbb{R}^n$, one gets a trapping region

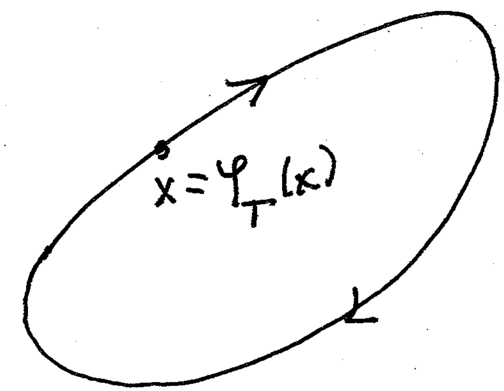


in which the vector field points in everywhere

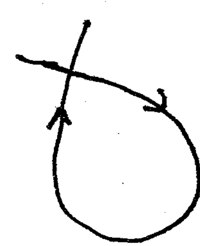
and then the flow exists for all forward time at least.

Stability of Periodic orbits

Recall $\varphi_{\mathbb{R}}(x)$ is a periodic orbit if there exists a $T>0$ so that $\varphi_t(x) \neq x$ for $0<t<T$ and $\varphi_T(x)=x$. The number T is called the period of $\varphi_{\mathbb{R}}(x) := \Gamma$

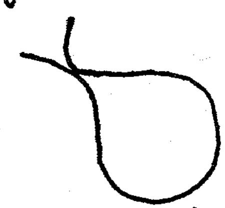


NOTE:



is not allowed

nor is

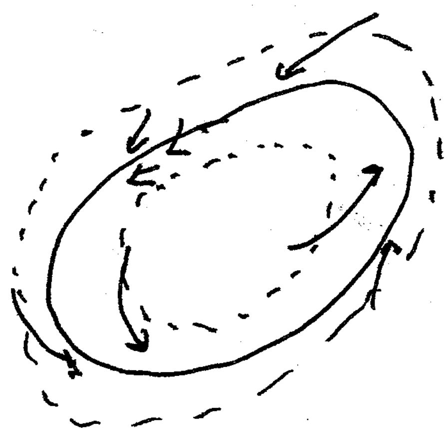


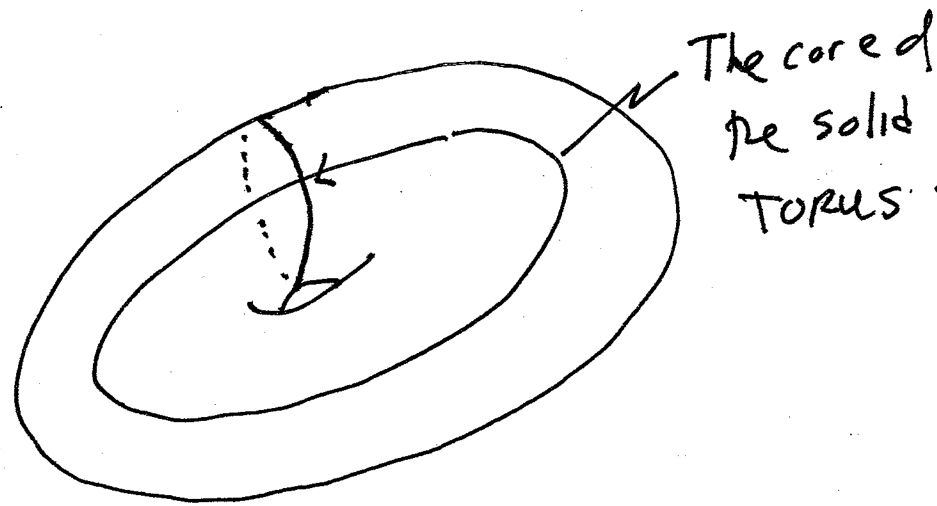
By existence and uniqueness

(note this is a single orbit, no rest point)

<u>DEF</u>: Γ is called <u>asymptotically stable</u> if $\exists$ a neighborhood U of Γ so that $x \in U \Rightarrow$

$$\varphi_t(x) \to \Gamma \text{ as } t \to \infty$$





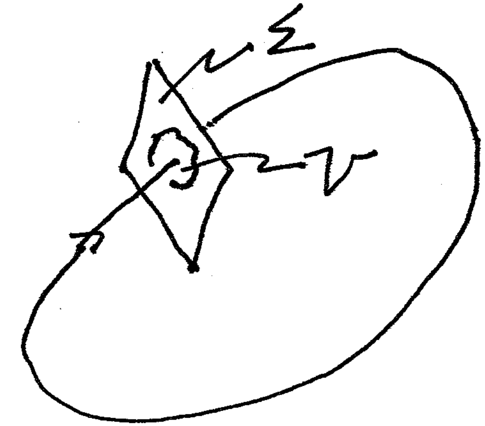
Γ is called <u>Stable</u> if for all U_2 exists U_1 with $\Gamma \subseteq U_1 \subseteq U_2$ so that $x \in U_1$

$$\Rightarrow \varphi_t(x) \in U_2 \text{ for all } t \geq 0$$

- Γ is called <u>unstable</u> if it is not stable.

- How do we compute stability?

DEF: Σ is a local cross section to Γ if $\exists$ nbd $V \subseteq \Sigma$ with $\Gamma \cap \Sigma \subseteq V$ and a continuous function $I: V \to (0,\infty)$ with $\varphi_t(x) \notin \Sigma$ $0 < t < I(x)$ and $\varphi_{I(x)}(x) \in \Sigma$.



$r: V \to \Sigma$ is the return or Poincaré map. It is a homeomorphism onto its image.

- If $\varphi_t : \mathbb{R}^n \circlearrowleft$ with periodic orbit Γ and Σ a smoothly embedded open disk with $\Gamma \cap \Sigma \neq \emptyset$ and Σ is transverse to $\vec{F}$

i.e. $\vec{F}|_\Sigma$ all points to one direction

$\Longrightarrow \Sigma$ is a local cross section to Γ.

- Let $\underline{P} = \Sigma \cap \Gamma$, so if Γ has period T
$\varphi_T(\underline{P}) = \underline{P}$ and $r(\underline{P}) = \underline{P}$, i.e. $\underline{P}$ is a fixed point of the return map. r.

Theorem Let Γ be a periodic orbit of $\varphi_t : \mathbb{R}^n \mathbb{R}$ with cross section Σ and return map $r : \Sigma \to \Sigma$ and $P = \Sigma \cap \Gamma$ so $r(P) = P$

(a) If p is asymptotically stable as a fixed point of $r \Rightarrow \Gamma$ is asymptotically stable as a periodic orbit of φ_t

(b) Similarly when p is stable and unstable implies the same for Γ under φ_t

Proof Let $V \subseteq \Sigma$ be such that $r : V \to \Sigma$ is defined and let $I(x)$ be the return time of x to Σ so $r(x) = \varphi_{I(x)}(x)$. Let T be the period of Γ and make V small enough that $0 < I(x) < 2T$

(a) Now $\exists \delta_1$ with $B_{\delta_1}(p) \cap \Sigma \subseteq V$ and $x \in B_{\delta_1}(p)$

$\Rightarrow r^n(x) \to p$ as $n \to \infty$.

Given $\varepsilon > 0$

Now using the continuity of φ_t, $\exists \delta < \delta_1$ so that $x \in B_\delta(p) \cap \Sigma \Rightarrow d(\varphi_t(x), p) \le d(\varphi_t(x), \varphi_t(p)) < \varepsilon$ (1) for $0 < t < 2T$.

For $x \in B_\delta(p) \cap \Sigma$, let N be such that $r^n(x) \in B_\delta(p) \cap \Sigma$ for $n > N$

Let $T_n = \sum_{i=1}^{n} T(r^i(x))$ so $\varphi_{T_n}(x) = r^n(x)$

Thus $t > T_N$ implies $\varphi_{T_n}(x) \in B_\delta(p) \cap \Sigma$

and thus by (1) $d(\varphi_t(x), p) < \varepsilon$

This proves $x \in B_{\delta_1}(p) \cap \Sigma$ has $\varphi_t(x) \to p$ as $t \to \infty$

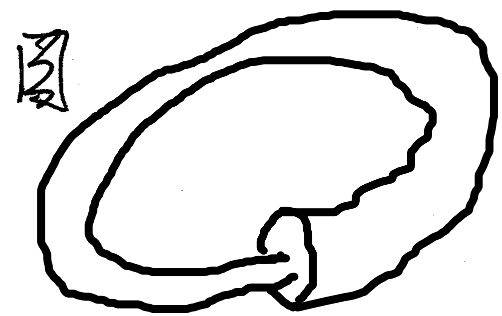
Now assume $\exists$ open $U \subseteq \Sigma$ with $p \in U$ and $r(U) \subseteq U$ (This follows from Hartman-Grobman if r is differentiable with eigenvalues Dr inside unit disk) and U is a topological disk.

Let $U' = \{\varphi_t(x) : x \in U,\ 0 < t \leq T(x)\}$

then $y \in U' \Rightarrow \varphi_t(y) \to \Gamma$ as $t \to \infty$

(b) is similar, but more technical

(c) is easy



- Now we have theorems on the stability of fixed points: example, $f(p)=p$, f is a diffeomorphism and $|\lambda|<1$ for all eigenvalues λ of $Df(p) \Rightarrow$
  p is a asymptotically stable fixed points.
- There is an analogous result for periodic orbits of flows using "Floquet multipliers"
- Assume now $\varphi_t(x)$ is twice differentiable in both x and t so we can ignore some technical issues
- $D\varphi_t(x)$ is the space derivative i.e. derivative w.r.t. x

  $\frac{\partial \varphi_t(x)}{\partial t}$ is the time derivative

The ODE is

$$\frac{\partial \varphi_t(x)}{\partial t} = \vec{F}(\varphi_t(x))$$

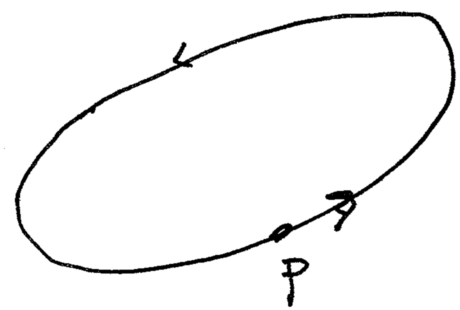
Taking the space derivative and switch order

$$\frac{\partial D\varphi_t(x)}{\partial t} = D\frac{\partial \varphi_t(x)}{\partial t} = D\vec{F}(\varphi_t(x)) \cdot D\varphi_t(x)$$

This is called the variational equation

Thus if I fix x and let $M(t) = D\varphi_t(x)$ and $A(t) = D\vec{F}(\varphi_t(x))$ then $M(t)$ satisfies the time-dependent matrix equation $\frac{dM}{dt}(t) = A(t)M(t)$

We apply this to a periodic orbit of period T



- Pick $p \in \Gamma$ and let: $M(t) = D\varphi_t(p)$

then the spectrum of $M(T)$ are the Flouqet multipliers

- Let $A(t) = D\vec{F}(\varphi_t(p))$, then $A(t+T) = A(t)$

 and $M(t)$ solves the periodic matrix equation

$$\frac{dM}{dt}(t) = A(t)\, M(t)$$

- We shall see that the Flouqet multipliers are (almost) the same as $Dr(p)$ for the return map at the fixed point