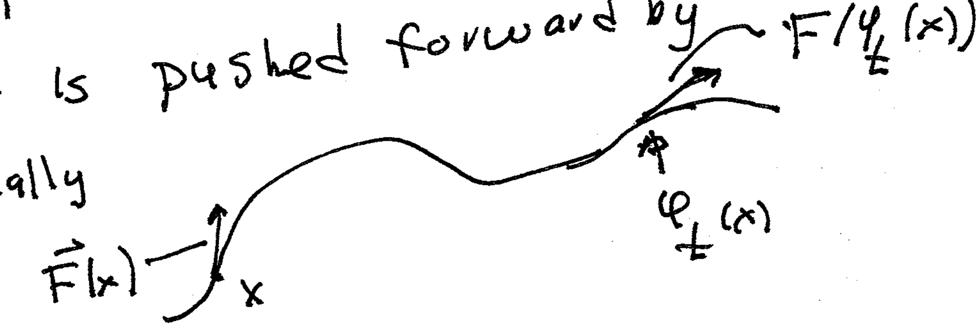
Stability of Periodic orbits, cont

- $\Gamma = \varphi_{\mathbb{R}}(p)$ is a periodic orbit with cross section Σ (a smoothy embedded disk) with return map $r : V \to \Sigma$ with fixed point p
- we assume $\varphi_t(x)$ is C^2 which implies that r is a diffeomorphism
- $D\varphi_t(x)$ is the space derivative, i.e. wrt $x \in \mathbb{R}^n$
- $\dfrac{\partial \varphi_t}{\partial t}$ is the time derivative

- Assume Γ has period T

<u>Theorem</u> (1) For any point x, $D\varphi_t(x)[\vec{F}(x)] = \vec{F}(\varphi_t(x))$

This says the vector field is pushed forward by the flow infinitesimally



(2) On the periodic orbit, 1 is always an eigenvalue of $D\varphi_T(p)$ with eigenvector $\vec{F}(p)$

(3) If q is another point of Γ $\Rightarrow$ $D\varphi_T(q)$ and $D\varphi_T(p)$ are conjugate as matrices and thus have the same spectrum

PROOF:
For (a)

$$\left.\frac{\partial(\varphi_t \circ \varphi_s(x))}{\partial s}\right|_{s=0} = \left.\frac{\partial \varphi(\varphi_s(x), t)}{\partial s}\right|_{s=0}$$

$$= D\varphi(\varphi_s(x), t) \cdot \left.\frac{\partial \varphi_s(x)}{\partial s}\right|_{s=0}$$

$$= D\varphi_t(x) \cdot \vec{F}(x)$$

on the other hand $\left.\frac{\partial(\varphi_t \circ \varphi_s(x))}{\partial s}\right|_{s=0} = \left.\frac{\partial \varphi_s(x)}{\partial s}\right|_{s=t}$

$$= \vec{F}(\varphi_t(x))$$

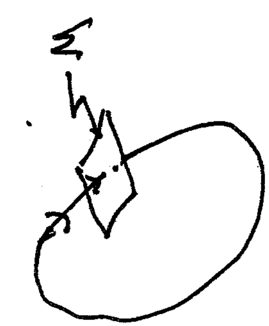
where twice we used the fact the $\varphi_t(x)$ solves the DE.

For (b) $\varphi_T(p)=p$ so using (a)

$$\vec{F}(p) = \vec{F}(\varphi_T(p)) = D\vec{F}(\varphi_T(p))[\vec{F}(p)]$$

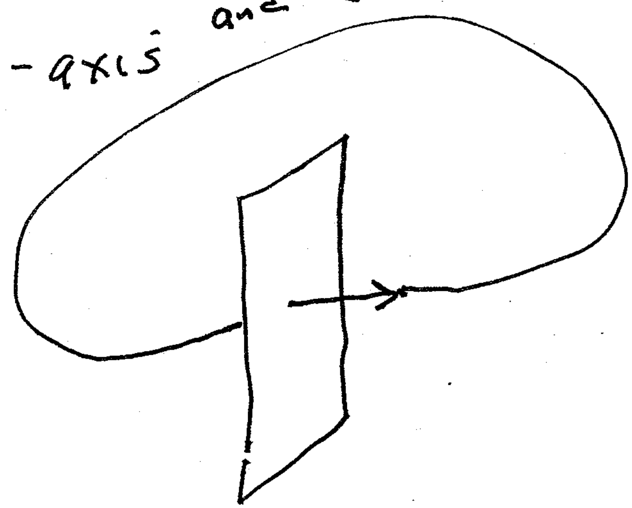
(c) Home work. $\boxed{}$

<u>DEF:</u> If $\operatorname{spec}(D\varphi_T(p)) = \{1, \lambda_1, .., \lambda_{n-1}\}$ then $\{\lambda_1, .., \lambda_{n-1}\}$ are the <u>characteristic</u> or <u>Floquet multipliers</u> of Γ.



<u>Theorem</u>: With the hypothesis of above $\operatorname{Spec}(Dr(p))$ = the Floquet multiplies of Γ.

Sketch proof: By changing coordinates we may assume that $p = \vec{0}$, Σ is a disk in $S = \{(0,y) : y \in \mathbb{R}^{n-1}\}$ and $\vec{F}(0)$ points in the direction of the x-axis and $I(x) \geq 1$ for $x \in \Sigma$.



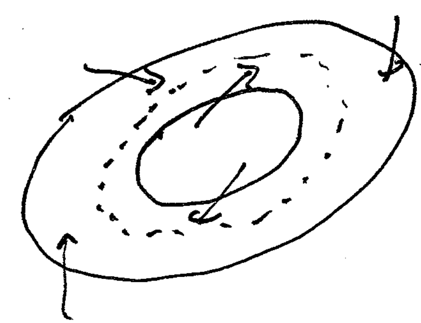
Then $D\varphi_T(\vec{0})$ splits as $A \oplus B$ with $A = D\varphi_T(\vec{0})$ restricted to S and $B = 1$ (since $D\varphi_T(\vec{0})[\vec{F}(\vec{0})] = \vec{F}(\vec{0})$). Thus $A = Dr(\vec{0})$ as needed (lots to be justified there)

<u>CORR</u>: • If Γ has all Floquet multipliers with $|\lambda|<1 \Rightarrow \Gamma$ is asymptotically stable

• If $\exists$ a Floquet multiplier λ with $|\lambda|>1$ $\Rightarrow \Gamma$ is unstable.

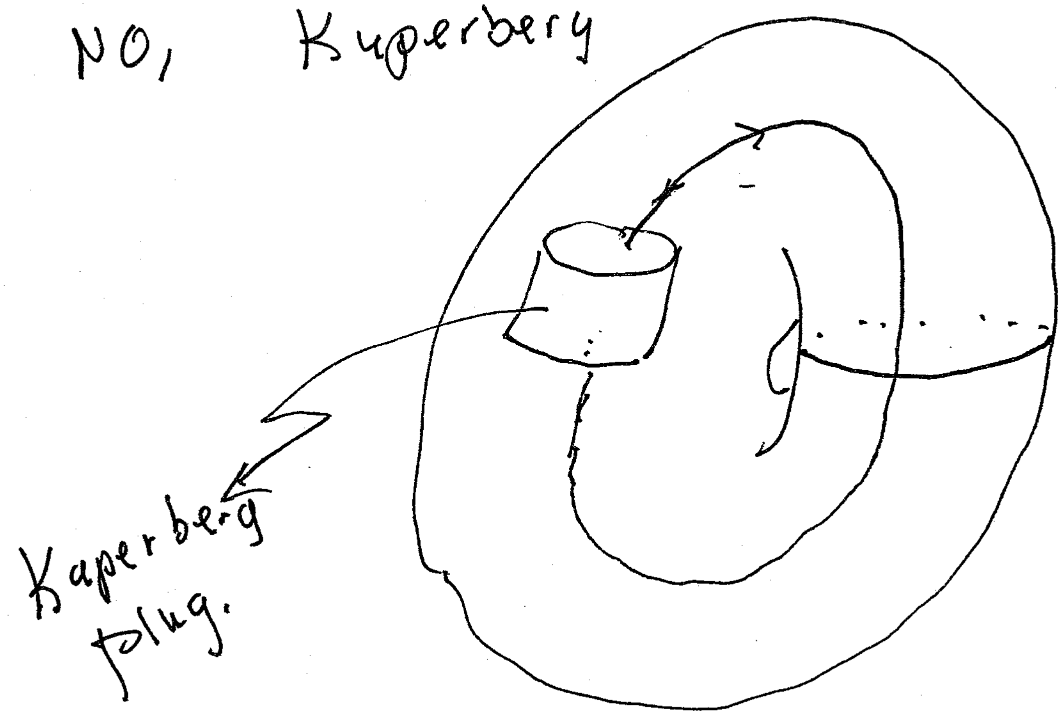
A Remark on the plane vs higher dimension:

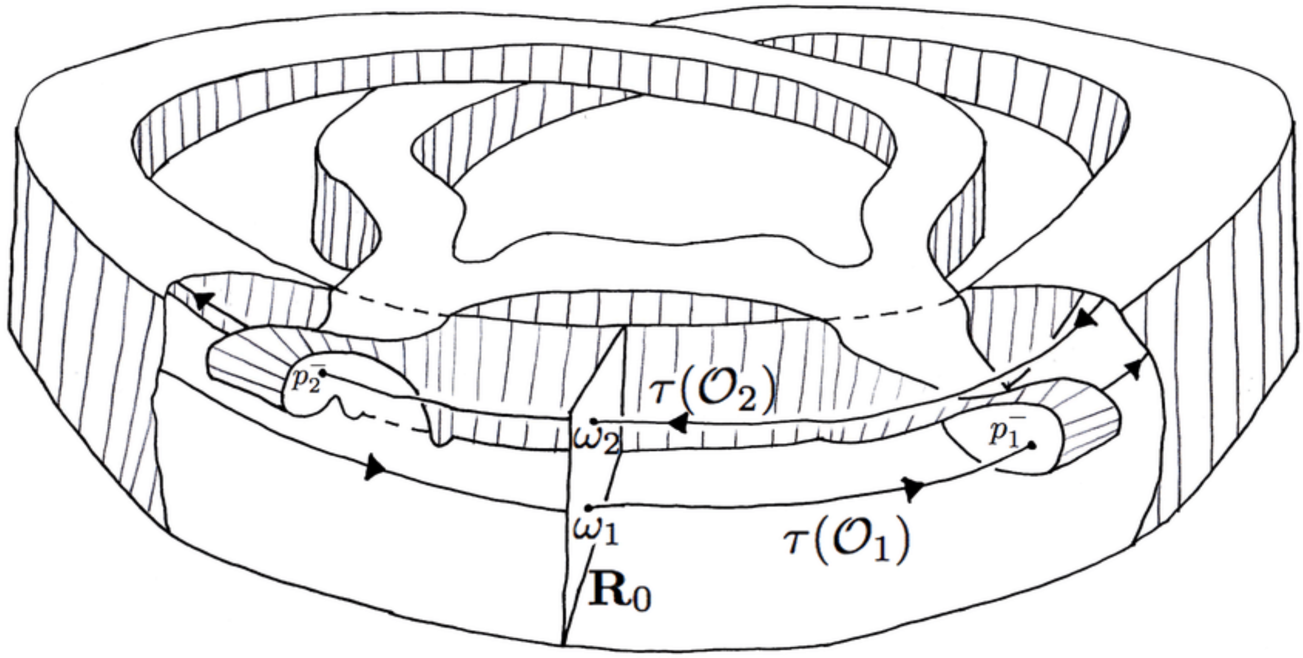
Poincaré-Bendixson: If $\varphi_t : \mathbb{R}^2$ is a C^1 flow and D is a positively invariant region which contains no rest points $\Rightarrow \varphi_t$ has a periodic orbit in D.



NOT TRUE in $\mathbb{R}^3$ SAY have positively invariant solid torus $\Rightarrow$ must contain a periodic orbit??
with no rest points inside

NO, Kuperberg




p_2^-
$\tau(\mathcal{O}_2)$
ω_2
p_1^-
ω_1
$\tau(\mathcal{O}_1)$
$\mathbf{R}_0$

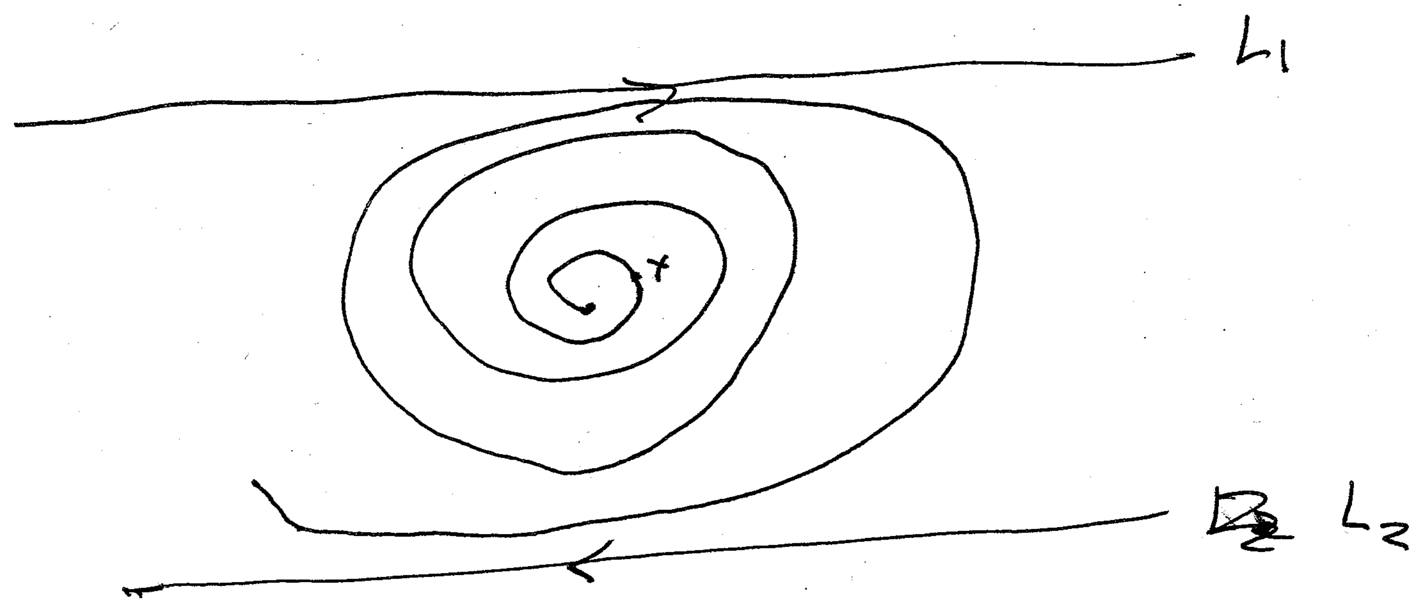
Before studying the stability of equilibrium we note and importance difference between discrete and continuous dynamics.

Theorem: Let φ_t be a continuous flow on a metric space X. Recall $\omega(x) = \{z : \exists t_k \to \infty \ \varphi_{t_k}(x) \to z\}$. If $O^+(x, \varphi_t)$ is contained in some compact subset of $X \Rightarrow \omega(x)$ is nonempty, compact and connected.

Proof: $\omega(x) = \bigcap_{T \geq 0} cl\left(\bigcup_{t \geq T} \{\varphi_t(x)\}\right)$ (HW)

each $\bigcup_{t \geq T} \{\varphi_t(x)\}$ is connected, $cl\left(\bigcup_{t \geq T} \{\varphi_t(x)\}\right)$ is compact (since $O^+(x, \varphi_t)$ is in some compact subset of X) and connected $\Rightarrow \bigcap_{T \geq 0} cl\left(\bigcup_{t \geq T} \{\varphi_t(x)\}\right)$ is nested intersection of cpt. conn. sets and so is cpt, conn. Rest is easy (HW).

Example: If $O^+(x, \varphi_t)$ is not bounded, $\omega(x)$ could be disconnected.



$$\omega(x) = L_1 \cup L_2$$

Stability of rest points

Assume $\varphi_t(x)$ is C^2 with associated vector Field $\vec{F}(x)$. Recall p is a rest point (equilibrium, singularity, ...) if $\varphi_t(p) = p \; \forall t \in \mathbb{R}$ which happens $\Longleftrightarrow$ $\vec{F}(p) = 0$

Summary of linear theory: $\frac{dx}{dt} = Ax$ A is $n \times n$ matrix with eigenvalues λ

(a) $\forall \lambda$, $Re(\lambda) < 0$, then $\vec{0}$ is [asymptotically] stable under φ_t soln flow.

(b) If $\exists \lambda$ with $Re(\lambda) > 0 \Rightarrow$ unstable

(c) If $\mathrm{spec}(A) \wedge \{Re(z) = 0\} \Rightarrow \vec{0}$ is hyperbolic

(d) If $\vec{0}$ is hyperbolic and $\exists \lambda_1$, $Re(\lambda_1) < 0$ and $\exists \lambda_2$, $Re(\lambda_2) > 0 \Rightarrow \vec{0}$ is a saddle

Summary of nonlinear theory $\quad \vec{F}(p) = 0$

and let $A = D\vec{F}(p)$ have eigenvalues λ

(a) If $Re(\lambda) < 0 \quad \forall \lambda \Rightarrow p$ is asymptotically stable

(b) If $\exists \lambda$ with $Re(\lambda) > 0 \Rightarrow p$ is unstable

(c) If $spec(A) \wedge \{Re(z) = 0\} = \phi$

$\Rightarrow p$ called hyperbolic

(d) If $\exists \lambda_1$ with $Re(\lambda_1) < 0$ and $\exists \lambda_2$ with $Re(\lambda_2) > 0$ and p is hyperbolic $\Rightarrow$

p is a saddle

- Proof via linear theory and Hartman - Grobman

Remark: $Re(\lambda) < 0 \not\Rightarrow |e^{\lambda}| < 1$

Example

Let $\vec{F}(x,y) = (y,\ x - x^3 - y)$

Find all rest points and classify them

SOLN $\vec{F}(x,y) = \vec{0} \Rightarrow y = 0$ and $x - x^3 = 0$ so $x = 0, \pm 1$

$$DF = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & -1 \end{bmatrix} \qquad \lambda = \frac{-1 \pm \sqrt{1 + 4(1-3x^2)}}{2}$$

at $(0,0)$ $\quad \lambda = -\frac{1 \pm \sqrt{5}}{2}$ $\quad$ Real $\lambda_1 < 0$ $\lambda_2 > 0$ $\Rightarrow$ saddle

$(1,0)$

$(-1,0)$ $\quad \lambda = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm \sqrt{7}\, i}{2}$

$\mathrm{Re}(\lambda) < 0$, asymptotically stable