$\boxed{\text{Stability for rest points for flows, cont.}}$

- Assume φ_t is C^2 with vector field $\vec{F}$ on $\mathbb{R}^n$
- p is a rest point (equilibrium, singularity, ...)

  $\iff \varphi_t(p) = p, \ \forall t \in \mathbb{R} \iff \vec{F}(p) = 0$

- Taylor's theorem for $\vec{F}$ expanded about p

$$F(x) = \vec{F}(p) + DF(p)(x-p) + \text{h.o.t.}$$
$$= \underline{DF(p)}(x-p) + \underline{\text{h.o.t}}$$

  dominates

  if the linear term.

- Change coordinates if the linear term and $p = \vec{0}$, the local behaviour near p is governed by the linear ode $\frac{d\vec{x}}{dt} = A\vec{x}$ with $A = \underline{D\vec{F}(p)}$

- NOTE: The "Linear terms dominate" always when p is hyperbolic - ie. $\mathrm{Spec}(A) \cap \{\mathrm{Im}(z)=0\} = \emptyset$.

- So we study $\frac{d\vec{x}}{dt} = A\vec{x}$ (*)

- This is always solved by the matrix exponential

$$e^B = I + B + \frac{B^2}{2} + \dots + \frac{B^n}{n!} + \dots$$

which always converges.

- The soln to (*) is thus $e^{At} = I + At + \frac{(At)^2}{2} + \dots + \frac{(At)^n}{n!} + \dots$

and so $\frac{de^{At}}{dt} = Ae^{At}$ by term by term differentiation

- The IVP $\frac{d\vec{x}}{dt} = A\vec{x}$ $\quad \vec{x}(0) = \vec{x}_0$ has

soln $\quad x(t; x_0) = e^{At}\vec{x}_0$

- But, how do you compute e^{At}?

- Answ: Jordan form.

- method 1: Assume $A = CJC^{-1}$, J is simple Jordan form.

$$\Rightarrow \frac{d\vec{x}}{dt} = A\vec{x} = CJC^{-1}\vec{x}$$

$$\Rightarrow \frac{d\, C^{-1}\vec{x}}{dt} = JC^{-1}\vec{x}$$

Let $\vec{y} = C^{-1}\vec{x} \Rightarrow \boxed{\frac{d\vec{y}}{dt} = J\vec{y}}$

- <u>method 2</u>

$$e^{At} = e^{CJC^{-1}t} = I + CJC^{-1}t + \frac{(CJC^{-1})^2 t^2}{2} + \cdots + \frac{(CJC^{-1})^n t^n}{n!} + \cdots$$

$$\boxed{\text{but } (CJC^{-1})^n = CJ^nC^{-1}}$$

$$= C\left(I + Jt + \frac{J^2t^2}{2} + \cdots + \frac{J^n t^n}{n!} \cdots\right)C^{-1}$$

$$= C e^{Jt} C^{-1}$$

- so up to change of coordinates we just need to understand $\frac{d\vec{x}}{dt} = J\vec{x}$ where J is a Jordan form.

- The 2×2 case

(1) $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\quad e^{Jt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$

(2) $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. We need a Lemma:

$$\boxed{e^{A}e^{B} = e^{A+B} \Longleftrightarrow AB = BA}$$

$$e^{A}e^{B} = \left(I + A + \frac{A^2}{2} + \cdots\right)\left(I + B + \frac{B^2}{2} + \cdots\right)$$

$$= I + (A+B) + \frac{A^2}{2} + AB + \frac{B^2}{2} + \cdots$$

$$e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2} + \cdots$$

$$A^2 + 2AB + B^2$$
$$\parallel$$
$$A^2 + AB + BA + B^2$$

$\boxed{\text{requires } AB = BA}$

(5)

Now write $A = \lambda I + (A - \lambda I)$ note that $\boxed{A = J}$

$$(\lambda I)(A-\lambda I) = (A-\lambda I)\lambda I$$

and $(A-\lambda I)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

so $(A-\lambda I)^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $n \geq 2$.

Thus $e^{At} = \left(e^{\lambda I t}\right)\left(e^{(A-\lambda I)t}\right)$

$$= \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}\left[I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t + \cancel{\frac{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 t^2}{2}} + \cdots \right]$$

$$= \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

(3) $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ $\quad J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x - \beta y \\ \beta x + \alpha y \end{pmatrix}$

$\lambda = \alpha \pm i\beta$

$(\alpha + i\beta)(x+iy) = \alpha x - \beta y + i(\beta x + \alpha y)$

$\underline{\underline{\text{so}}}$ $\quad J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Leftrightarrow (\alpha+i\beta)(x+iy) = a + ib$

so we treat $J\begin{pmatrix} x \\ y \end{pmatrix}$ as $(\alpha + i\beta)\cdot z$

$$e^{Jt} \leftrightarrow e^{(\alpha+i\beta)t} = e^{\alpha t} e^{i\beta t} \equiv e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

$$\leftrightarrow e^{\alpha t}\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}$$

(or compute $e^{Jt} = I + Jt + \frac{(Jt)^2}{2} + \dots$ by real or complex methods)

Dynamics:

(1) $\frac{d\vec{x}}{dt} = J\vec{x}$ $\quad \vec{x}(0) = \vec{x}_0$. $\quad J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$x(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \vec{x}_0$

$\lambda_1, \lambda_2 < 0 \Rightarrow$ sink, stable

$\lambda_1 < 0$ $\lambda_2 > 0 \Rightarrow$ saddle unstable

$\lambda_1, \lambda_2 > 0 \Rightarrow$ source, unstable

one $\lambda_L = 0$, $e^{\lambda_L t} = 1$, so no motion in that coord.

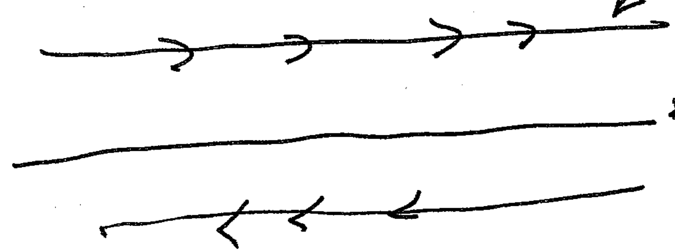
(2) $\frac{dx}{dt} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \vec{x} \Rightarrow x(t) = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \vec{x}_0$

$\lambda < 0 \Rightarrow t e^{\lambda t} \to 0$, so sink, stable

$\lambda > 0 \Rightarrow$ source, unstable

$\lambda = 0 \Rightarrow x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \vec{x}_0$ shear flow. unstable

$x(t) = \begin{pmatrix} x_0 + t y_0 \\ x_0 \end{pmatrix}$ $\qquad \begin{aligned} \dot{x} &= y \\ \dot{y} &= 0 \end{aligned}$ is ODE



$\dot{x} = y$, velocity = height above x-axis

all fixed pts, "the wall"

(3) $\quad \dfrac{dx}{dt} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} x$, $\quad x(t) = e^{\alpha t} \underbrace{\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}}_{\text{rigid rotation at angular speed } \beta.} x_0$

$\alpha = \mathrm{Re}(\lambda) < 0 \Rightarrow$ stable, spiral sink

$\alpha = \mathrm{Re}(\lambda) > 0 \Rightarrow$ unstable, spiral source

$\alpha = \mathrm{Re}(\lambda) = 0 \Rightarrow$ center, stable, not asymp stable

- So $\lambda_i = 0$ or $\lambda = \alpha + i\beta$ with $\alpha = 0$ (or, $\lambda \in \{Re(z)=0\}$) is the delicate case, In others one can decide easily stasity vs instability

- Rest point p is called hypersolic if $Spec(DF(\hat{p})) \cap \{Re(z)=0\} = \emptyset$.

- Hartman-Grobmand says that in the hypersolic case, up to change of coordinates, a nonlinear hypersolic rest point looks like its linearization

<u>HG Theorem</u> Assume φ_t is C^2 with vector field $\vec{F}$ and p is a hyperbolic rest point for φ_t. Let ψ_t be the soln flow to $\frac{d\vec{x}}{dt} = DF(p)\vec{x}$. There exists a neighborhood U of $\vec{0}$ and a homeomorphism $h: U \to V$ where V is an open set containing p so that

$$\begin{tikzcd} U \arrow[r, "\psi_t"] \arrow[d, "h"'] & U \arrow[d, "h"] \\ V \arrow[r, "\varphi_t"'] & V \end{tikzcd}$$

* Note this is a strong top. conj. as it preserves the time parameterization.

When $n > 2$ the Jordan form J splits into blocks

$$\begin{bmatrix} [\;] & & & & \\ & [\;] & & & \\ & & [\;] & & \\ & & & [\;] & \\ & & & & [\;] \end{bmatrix}$$

and so

e^{Jt} splits into invariant subspaces under the flow

The block $B = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$ satisfies $(B-\lambda I)^k = 0$ when

B is $k \times k$ so $e^{Bt} = e^{\lambda I t} e^{B - \lambda I t} =$

$$e^{\lambda I t}\left[I + (B - \lambda I t) + \frac{(B - \lambda I t)^2}{2} + \dots \frac{(B - \lambda I t)^{k-1}}{(k-1)!} \right]$$

Contributes terms like $e^{\lambda t} p(t)$ with $\deg(p) = k-1$