Dynamics of circle homeomorphisms - Poincaré + Denjoy

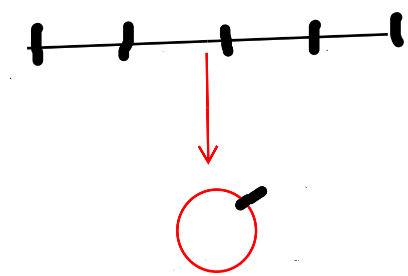
DS26

- Circle: $S^1 = \mathbb{R}/\mathbb{Z}$ or mod 1 (a)

  or $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ (b)

- The universal cover is $\mathbb{R}$. The covering map $\pi : \mathbb{R} \to S^1$ is either $\pi(\tilde{x}) = \tilde{x} \bmod 1$ (a)

  or $\pi(\tilde{x}) = e^{2\pi i \tilde{x}}$ (b)



- The covering translation (or deck translation) is $T : \mathbb{R} \to \mathbb{R}$, $T(\tilde{x}) = \tilde{x} + 1$

- NOTICE

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "T"] \arrow[dr, "\pi"'] & & \mathbb{R} \arrow[dl, "\pi"] \\ & S^1 & \end{tikzcd}$$

- Given $f: S^1 \circlearrowleft$ continuous, a lift is a map $\tilde{f}: \mathbb{R} \to \mathbb{R}$

  with
$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

- eg $f(\theta) = \theta + \alpha \mod 1$ then $\tilde{f}(\tilde{x}) = \tilde{x} + \alpha$ is a lift

- Since $f \circ \pi = \pi \circ \tilde{f}$, $\tilde{f}(T\tilde{x}) = T^k \tilde{f}(\tilde{x})$

  or $\tilde{f}(x+1) = \tilde{f}(x) + k$. k is called the degree of f (all lifts will have the same k)

- If $\tilde{f}$ is a lift of f, so is $T^n \tilde{f} = \tilde{f} + n$ for all n.

- If $\pi(\tilde{x}) = x$, $\tilde{x}$ is called a lift of x and $T^n \tilde{x} = \tilde{x} + n$ is also a lift for all n.

- For the topologist, degree $f = k \Rightarrow f_*: \pi_1(S^1) \circlearrowleft$ is $[\gamma] \mapsto k[\gamma]$

3

Recall f is continuous

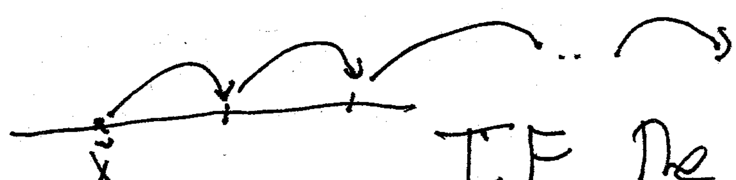
Lemma:

- Now assume $f: S^1 \circlearrowleft$ is an orientation preserving homeomorphism $\Leftrightarrow$ f is degree 1 and any lift $\tilde{f}$ is order preserving i.e.
$$\tilde{x}_1 < \tilde{x}_2 \Rightarrow \tilde{f}(\tilde{x}_1) < \tilde{f}(x_2)$$
Thus $\tilde{f}$ is an orientation preserving homeomorphism of $\mathbb{R}$

- Proof: HW.

- Let $\tilde{f}: \mathbb{R} \to \mathbb{R}$ be the lift of an orientation preserving homeomorphism f of S^1 ($f \in \mathrm{Homeo}_+(S^1)$) and a point $\tilde{x}$

Define the rotation number of $\tilde{f}$ is
$$\rho(\tilde{x}, \tilde{f}) = \lim_{n\to\infty} \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Delta(\tilde{f}^i(\tilde{x}))$$
where $\Delta(\tilde{x}) = \tilde{f}(\tilde{x}) - \tilde{x}$, the displacement so $\rho(\tilde{x}, \tilde{f})$ is average displacement



IF the Limit exists (which we will prove)

(1) <u>Remark</u>: $\tilde{f}(\tilde{x}) = \tilde{x} + \alpha \Rightarrow \tilde{f}^n(\tilde{x}) = \tilde{x} + n\alpha$

so $\rho(\tilde{x}, \tilde{f}) = \lim_{n\to\infty} \dfrac{\tilde{x} + n\alpha - \tilde{x}}{n} = \alpha$

(2) <u>Remark</u>:

$$\frac{\left| |\tilde{f}^n(\tilde{x})| - |\tilde{x}| \right|}{n} \le \frac{|\tilde{f}^n(\tilde{x}) - \tilde{x}|}{n} \le \frac{|\tilde{f}^n(\tilde{x})| + |\tilde{x}|}{n}$$

and so $\cancel{\lim_n}\ \rho(\tilde{x}, \tilde{f}) = \lim_{n\to\infty} \dfrac{\tilde{f}^n(\tilde{x})}{n}$, which is easier to use sometimes

(3) $\tilde{f}(\tilde{x}+1) = \tilde{f}(\tilde{x}) + 1$

Let $\varphi(\tilde{x}) = \tilde{f}(\tilde{x}) - \tilde{x}$ Thus $\varphi(\tilde{x}+1) = \tilde{f}(\tilde{x}+1) - (\tilde{x}+1)$

$= \tilde{f}(\tilde{x}) + 1 - (\tilde{x}+1) = \tilde{f}(\tilde{x}) - \tilde{x} = \varphi(\tilde{x})$. So φ is a continuous, periodic function so $\exists B$ with $|\varphi(\tilde{x})| < B$ $\forall \tilde{x} \in \mathbb{R}$

(4) By induction $\tilde{f}^n(\tilde{x}+1) = \tilde{f}^n(\tilde{x})+1$ and

$\tilde{f}^n(\tilde{x}+k) = \tilde{f}^n(\tilde{x})+k$, $\forall k \in \mathbb{Z}$

Thus $\tilde{f}^n(\tilde{x}+k) - (\tilde{x}+k) = \tilde{f}^n(\tilde{x}) - \tilde{x}$

Thus $\rho(\tilde{x}+k, \tilde{f}) = \rho(\tilde{x}, \tilde{f})$ i.e. the rotation number is ind. of the choice of lift of $x \in S^1$

Lemma If $\rho(\tilde{x}, \tilde{f})$ exists for some $\tilde{x} \in \mathbb{R}$ then $\forall \tilde{y} \in \mathbb{R}$, $\rho(\tilde{y}, \tilde{f})$ exists and $\rho(\tilde{x}, \tilde{f}) = \rho(\tilde{y}, \tilde{f})$. So ROT # is independent of the choice of point in $\mathbb{R}$.

Proof By Remark (4) we can assume $\tilde{x}, \tilde{y} \in [0,1)$ and also by Remark 4, $\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y}) \in \tilde{f}^n[0,1)$

$= [\tilde{f}^n(0), \tilde{f}^n(1)) = [\tilde{f}^n(0), \tilde{f}^n(0)+1)$ using the fact that $\tilde{f}$ is an orientation preserving homeomorph. and so

$$|\tilde{f}^n(\tilde{x}) - \tilde{f}^n(\tilde{y})| < 1 \quad \text{and so}$$

Thus

$$\lim_{n \to \infty} \left| \frac{\tilde{f}^n(\tilde{x})}{n} - \frac{\tilde{f}^n(\tilde{y})}{n} \right| = 0 \quad \text{and so}$$

$\rho(\tilde{x}, \tilde{f}) = \rho(\tilde{y}, \tilde{f})$ using Remark(2).

Theorem $\rho(\tilde{x},\tilde{f})$ always exists. It is rational when f has a periodic point

Proof There are two cases. First assume f has a periodic point x_0, so $f^q(x_0)=x_0$. Thus for $\tilde{x}_0$ with $\pi(\tilde{x}_0)=x_0$ we have $\tilde{f}^q(\tilde{x}_0)=\tilde{x}_0+p$ for some $p\in\mathbb{Z}$, then by induction, $\tilde{f}^{kq}(\tilde{x}_0)=\tilde{x}_0+kp$.

Now for $n\in\mathbb{N}$, write $n=kq+r$ with $0\le r<q$

Thus $\tilde{f}^n(\tilde{x}_0)=\tilde{f}^r(\tilde{f}^{kq}(\tilde{x}_0))=\tilde{f}^r(\tilde{x}_0+kp)=\tilde{f}^r(\tilde{x}_0)+kp.$

Apply Remark 4 to $\tilde{f}^r(\tilde{x})-\tilde{x}$ for $0\le r<q$, we may find an M so that $|\tilde{f}^r(\tilde{x}_0)-\tilde{x}_0|<M$ for $0\le r<q$.

Thus $\lim_{n\to\infty}\left|\frac{\tilde{f}^n(\tilde{x}_0)-\tilde{x}_0}{n}\right| = \lim_{n\to\infty}\left|\frac{\tilde{f}^r(\tilde{x}_0)-\tilde{x}_0+kp}{n}\right|$

$$\leq \frac{M}{n} + \lim_{k\to\infty}\frac{kp}{kq+r} = \frac{p}{q} \quad \left(\text{independent of } 0 \leq r < q\right)$$

Using the previous lemma, this proves it when there is a periodic orbit

Now Assume f has no periodic orbits.

This implies that $\forall \tilde{x} \in \mathbb{R}$, $\tilde{f}^n(\tilde{x})-\tilde{x} \notin \mathbb{Z}$ $\forall n \neq 0$.

Thus $\forall n \; \exists k_n \in \mathbb{Z} \; \forall \tilde{x} \in \mathbb{R}$

$$k_n < \tilde{f}^n(\tilde{x}) - \tilde{x} < k_n + 1 \quad (*)$$

Now we just need $\rho(\tilde{x}, \tilde{f})$ existing for one point so we use $\tilde{x} = 0$. Thus

$$k_n < \tilde{f}^n(0) - 0 < k_n + 1$$

Now $(*)$ holds $\forall \tilde{x} \in \mathbb{R}$, so it also holds for $\tilde{f}^n(0)$

and so $k_n < \tilde{f}^{2n}(0) - \tilde{f}^n(0) < k_n + 1$

continuing m steps

$$k_n < \tilde{f}^{mn}(0) - \tilde{f}^{(m-1)n}(0) < k_n + 1$$

Summing and cancelling

$$m k_n < \tilde{f}^{mn}(0) < m(k_n + 1)$$

Thus $\dfrac{k_n}{n} < \dfrac{\tilde{f}^{mn}(0)}{mn} < \dfrac{k_n+1}{n}$ (2)

but above we had $k_n < \tilde{f}^n(0) < k_{n+1}$

so $\dfrac{k_n}{n} < \dfrac{\tilde{f}^n(0)}{n} < \dfrac{k_n+1}{n}$ (3)

combining (2) and (3)

$$\left| \frac{\tilde{f}^{mn}(0)}{mn} - \frac{\tilde{f}^n(0)}{n} \right| < \frac{1}{n} \quad (4)$$

we do the same argument interchanging the roles of m and n

$$\left| \frac{\tilde{f}^{mn}(0)}{mn} - \frac{\tilde{f}^m(0)}{m} \right| < 1/m \quad (5)$$

Combining (4) and (5)

$$\left| \frac{\tilde{f}^n(0)}{n} - \frac{\tilde{f}^m(0)}{m} \right| < \frac{1}{n} + \frac{1}{m}$$

Thus considering the sequence $\left\{ \frac{\tilde{f}^k(0)}{k} \right\}$ given $\varepsilon > 0$ we can find K so that $m, n > K$ implies

$$d\left(\frac{\tilde{f}^n(0)}{n}, \frac{\tilde{f}^m(0)}{m} \right) < \varepsilon$$

and so $\left\{ \frac{\tilde{f}^k(0)}{k} \right\}$ is Cauchy and Thus converges and by Remark (2) so does $\rho(0, \tilde{f})$ $\blacksquare$

- So we know that for $\tilde{f}$ a lift of a orientation pres. homeomorphism $f: S^1 \to S^1$, $\rho(\tilde{x}, \tilde{f})$ exists and is independent of $\tilde{x}$, so we just write $\rho(\tilde{f})$.

- Notice that $\rho(\tilde{f})$ is associated with $\tilde{f}$ not f.

- If $\tilde{f}_1$ is another lift of f, then $\tilde{f}_1 = \tilde{f} + k$ for some k, and then it is easy to show that

$$\rho(\tilde{f}_1) = \rho(\tilde{f}) + k$$

- Thus we <u>define</u> $\rho(f) = \Pi(\rho(\tilde{f})) = \rho(\tilde{f}) \bmod 1$ where $\tilde{f}$ is any lift of f. This is independent of choice of lift.

- <u>example</u> $\tilde{f}(\tilde{x}) = \tilde{x} - \frac{\sqrt{2}}{2} \Rightarrow \rho(f) = 1 - \frac{\sqrt{2}}{2}$

  and notice that rotating S^1 counter clockwise by $1 - \frac{\sqrt{2}}{2}$ is the same as rotating it clockwise by $\frac{\sqrt{2}}{2}$.

- <u>Def:</u> The lift $\tilde{f}$ with $\rho(\tilde{f}) \in [0,1)$ is called the <u>preferred lift</u> so $\rho(\tilde{f}) = \rho(f)$