Dynamics of Circle homeomorphism, continued

- We know that for any orientation preserving $f : S^1 \circlearrowleft$ its "average rate of rotation" always exists, $\rho(f)$.
- How much does $\rho(f)$ tell you about the dynamics of f? (a fair amount)
- Theorem (Poincaré) $f : S^1 \circlearrowleft$ is o.p. homeomorphism

(a) $\rho(f) = p/q \in \mathbb{Q} \iff f$ has a periodic point

(b) if $\rho(f) = \underline{\alpha} \notin \mathbb{Q} \Rightarrow \exists$ "nice" $g : S^1 \to S^1$

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "g"'] & S^1 \arrow[d, "g"] \\ S^1 \arrow[r, "R_\alpha"] & S^1 \end{tikzcd} \qquad R_\alpha(\theta) = \theta + \underline{\alpha}.$$

i.e. semiconjugate the rigid ROTATION by its ROT. #.

- Recall that the equivalence relation (isomorphism) is the class of dynamical systems is topological conjugacy.

- So top. conj. circle homeomorphisms should have the same Rotation number.

- **Lemma**: If f and h are orientation preserving homeomorphisms of $S^1 \Rightarrow \rho(f) = \rho(hfh^{-1})$

Proof Let $\tilde{f}, \tilde{h}$ be lifts and so $\tilde{h}\tilde{f}\tilde{h}^{-1}$ is a lift of hfh^{-1}!

Then $\dfrac{(\tilde{h}\tilde{f}\tilde{h}^{-1})^n(\tilde{x})-\tilde{x}}{n} = \dfrac{\tilde{h}\tilde{f}^n\tilde{h}^{-1}(\tilde{x})-\tilde{x}}{n}$

$$= \frac{\tilde{h}(\tilde{f}^n\tilde{h}^{-1}(\tilde{x})) - \tilde{f}^n\tilde{h}^{-1}(\tilde{x})}{n} + \frac{\tilde{f}^n\tilde{h}^{-1}(\tilde{x}) - \tilde{h}^{-1}(\tilde{x})}{n}$$

$$+ \frac{\tilde{h}^{-1}(\tilde{x})-\tilde{x}}{n}$$

Now recall from last lecture, since $\tilde{h}$ is a lift of an o.p. circle homeo, $\exists M_1$ so that $|\tilde{h}(\tilde{y})-\tilde{y}| < M_1$, $\forall \tilde{y}\in\mathbb{R}$ and similarly $\exists M_2$ so that $|\tilde{h}^{-1}(\tilde{y})-\tilde{y}| < M_2$, $\forall \tilde{y}\in\mathbb{R}$

Thus $\left|\dfrac{\tilde{h}(\tilde{f}^n(\tilde{h}^{-1}(\tilde{x}))) - \tilde{f}^n\tilde{h}^{-1}(\tilde{x})}{n}\right| < \dfrac{M_1}{n} \to 0$

and similary $\left|\dfrac{\tilde{h}^{-1}(\tilde{x})-\tilde{x}}{n}\right| < \dfrac{M_2}{n} \to 0$

Thus $\lim_{n\to\infty} \frac{(\tilde{h}\tilde{f}\tilde{h}^{-1})^n(\tilde{x}) - \tilde{x}}{n}$ $[= \rho(\tilde{h}\tilde{f}\tilde{h}^{-1})]$

$$= \lim_{n\to\infty} \frac{\tilde{f}^n(\tilde{h}^{-1}(\tilde{x})) - \tilde{h}^{-1}(\tilde{x})}{n}$$

$$= \lim_{n\to\infty} \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n}$$ [since we have shown ROT is ind of point]

$$= \rho(\tilde{f})$$

$$\Rightarrow \rho(hfh^{-1}) = \rho(f) \quad \blacksquare$$

First part of Poincaré's Theorem.

(1) $\rho(f) = p/q \Leftrightarrow f$ has a periodic orbit

<u>Proof</u> ($\cancel{\Leftrightarrow}$) WAS done already

($\Longrightarrow$) . We assume that $\rho(f) = p/q$ but f does not have a periodic point. Now we showed above that if f has a period q point then $\rho(f) = p/q$ for some p. Thus it suffices to assume that $\rho(f) = p/q$ and f has no period q point since it can't have a period k point unless $k | q$.

Now it is easy to check that $\underline{\rho(\tilde{f}^m) = m\rho(\tilde{f})}$

and so $\rho(\tilde{f}^q) = p$. Thus we may find another lift $\tilde{g}$ of $\tilde{f}^q$ so that $\tilde{g}$ $\underline{\text{has no fixed points}}$ and $\underline{\rho(\tilde{g}) = 0}$.

Now using the continuity of $\tilde{g}$ either $\tilde{g}(\tilde{x}) > \tilde{x}$ $\forall \tilde{x} \in \mathbb{R}$ or $\tilde{g}(\tilde{x}) < \tilde{x}$ $\forall \tilde{x} \in \mathbb{R}$. Assume the former, the latter is similar.

Now say $\exists\ k>0$ with $\tilde{g}^k(0)>1 \Rightarrow \tilde{g}^k(\tilde{g}^k(0)) > \tilde{g}^k(1)$

$= \tilde{g}^k(0)+1 > 2$ continuing, $\tilde{g}^{mk}(0) > m$ so

$\therefore\ \rho(\tilde{g}^k(0)) = \lim\limits_{m\to\infty} \dfrac{(\tilde{g}^k)^m(0)}{m} > 1$

so $\rho(\tilde{g}(0)) > 1/k$ by the initial remark, A contradiction.

So we may assume $\tilde{g}^n(0)<1$ for all n. Now using the fact that $\tilde{g}(\tilde{x})>\tilde{x}$, $\forall \tilde{x}\in\mathbb{R}$, $0<\tilde{g}(0)<\tilde{g}^2(0)<\ldots<1$

So $\{\tilde{g}^n(0)\}$ is a bounded, increasing sequence so $\exists\ p$

with $\tilde{g}^n(0)\to p$ and so $\tilde{g}^{n+1}(0)\to\tilde{g}(p)$

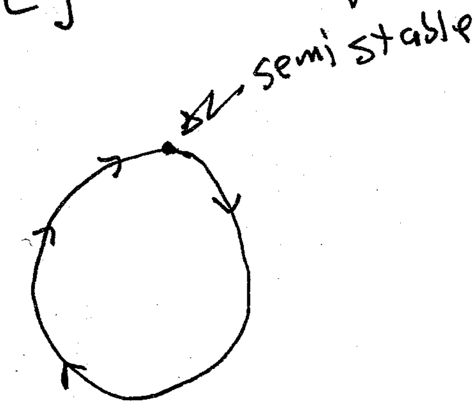
$\lim \tilde{g}^{n+1}(0) = \lim \tilde{g}^n(0) = p$ so

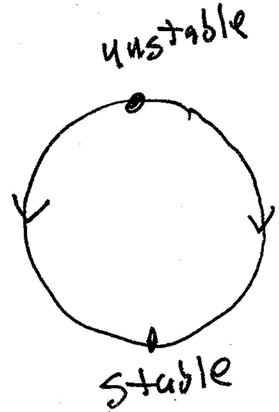
and

$\tilde{g}(p)=p$, a fixed point and a contradiction. $\blacksquare$

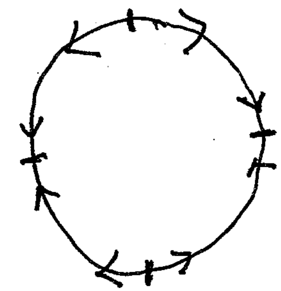
If $\rho(f) = p/q$, f can have a variety of dynamics

eg with $\rho(f) = 0$





north pole
south pole flow



Two sink–source pairs.

when $\rho(f) = p/q$, these type of examples are rotated.

We now explore the case when $\rho(f) \notin \mathbb{Q}$. The first Theorem explores the recurrent dynamics of f

We prove the second part of Poincare's Theorem next lecture.

Theorem Assume $f: S^1 \circlearrowleft$ is an orientation preserving homeomorphism with $\rho(f) \notin \mathbb{Q}$

(1) $\forall x, y \in S^1$, $\omega(x) = \omega(y)$. Call this set Σ

completely invariant

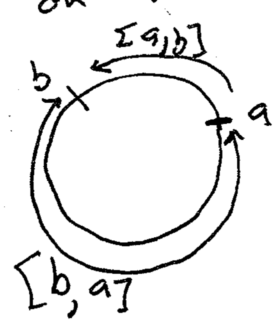
(2) Σ is perfect; and $f|\Sigma$ is a minimal set

(3) Either $\Sigma = S^1$ or Σ is perfect, nowhere dense and so is a CANTOR SET.

We need a lemma.

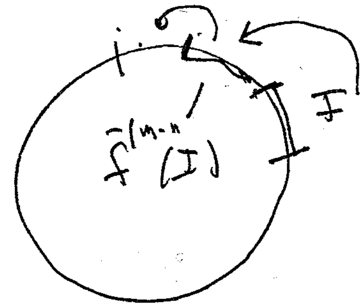
<u>Lemma</u> under hypothesis of the theorem, given $x \in S^1$ and $m > n$ $\Rightarrow$ $\forall y \in S^1$ $\exists$ $k > 0$ so that $f^k(y) \in [f^m(x), f^n(x)] := I$, i.e. the forward orbit of y hits I.

<u>Remark</u> Intervals on the circle are considered counterclockw



<u>Proof:</u> Consider $f^{-(m-n)}(I) = [f^n(x), f^{-m+2n}(x)]$

$f^{-2(m-n)}(I) = [f^{-m+2n}(x), f^{-2m+3n}]$ etc. We see that consecutive iterates of $f^{-(m-n)}$ acting on I abut, i.e. share an endpoint.

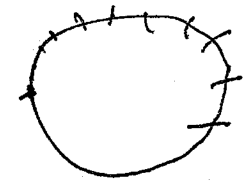


Now if $S' = \bigcup_{i=0}^{\infty} f^{-i(m-n)}(I)$

then $y \in S' \Rightarrow y \in f^{-i_0(m-n)}(I)$ some i_0 so

$f^{i_0(m-n)}(y) \in I$, and we are done.

So assume $S' \neq \bigcup_{i=0}^{\infty} f^{-i(m-n)}(I)$



This implies that $\{ f^{-i(m-n)}(f^n(x)) \}$ is monotonic

and bounded ons' so $\exists$ p with $f^{-i(m-n)}(f^n(x)) \to p$

and repeating a previous argument, p is a

fixed point for $f^{(m-n)}$ so $\rho(f^{(m-n)}) = 0$

so $\rho(f) = 0$, a contradiction [recall $\rho(f^k) = k\rho(f)$]

∎

Proof of Theorem

(a) Fix $x, y \in S^1$. Assume $x_0 \in \omega(x)$ so $\exists a_n \to \infty$ with $f^{a_n}(x) \to x_0$. Using the Lemma with $I = [f^{a_{n-1}}(x), f^{a_n}(x)]$ find b_n with $f^{b_n}(y) \in I$.

This implies $\lim\limits_{n\to\infty} f^{b_n}(y) = \lim\limits_{n\to\infty} f^{a_n}(x) = x_0$

Thus $x_0 \in \omega(y)$ and so $\omega(x) \subseteq \omega(y)$. Reversing the roles of x and y, $\omega(y) \subseteq \omega(x)$ so $\omega(x) = \omega(y)$.

(b) Pick $z \in \overline{X}$, by part (a), $\omega(z) = \overline{X}$ Thus $\overline{X}$ is minimal and so every point is a limit point. We proved previously that any omega limit set of a homeomorphism is compact and completely invariant.

(c) Assume $X \neq S^1$, Let $Fr(X)$ be the frontier of X ($= \{y : \forall \varepsilon, B_\varepsilon(y)$ contains points from both X and $S^1 - X\}$). It is easy to show that $Fr(X)$ is invariant and compact.

but on the other hand $z \in Fr(X)$ also has

$$\omega(z) = X$$ and since $Fr(X)$ is invariant

$$X = \omega(z) \subseteq Fr(X)$$, Thus $X = Fr(X)$

which implies it is nowhere dense.

In one dimension, This implies X is completely disconnected (it contains no intervals). Thus X is cpt, perfect and completely disconnected

So it is a Cantor set.