Orientation preserving homeomorphisms of S^1, cont.

- Last time we showed that if $\rho(f) = \alpha \notin \mathbb{Q}$ then $\exists !$ minimal set $\overline{X}$ and $\omega(x) = \overline{X}$, $\forall x \in S^1$ (and also $\alpha(x) = \overline{X}$, $\forall x \in S^1$ by considering f^{-1})

- The simplest case is when $\overline{X} = S^1$ and so f is transitive and the theorem below says that f is topologically conjugate to (S^1, R_α) where $R_\alpha(\theta) = \theta + \alpha$

- When $\overline{X} \subsetneq S^1$ we showed $\overline{X}$ is a Cantor set

- We give an example (These are called Denjoy examples) informally first.

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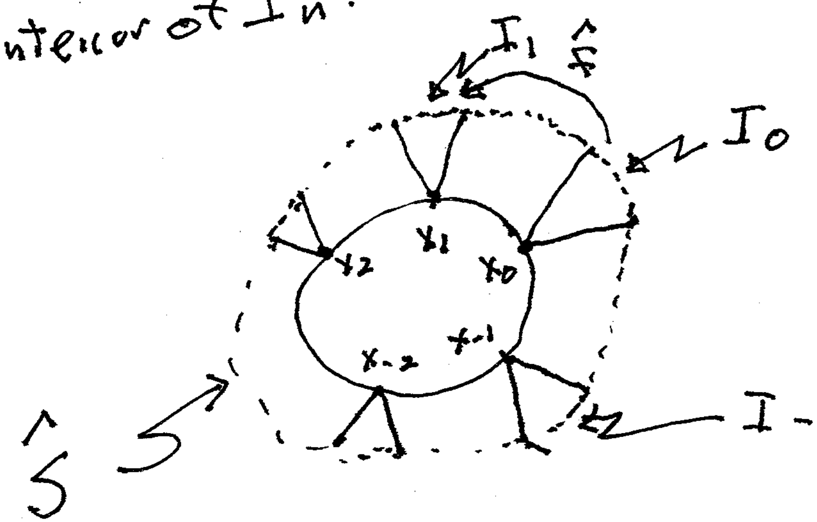
Pick a sequence a_n with $n \in \mathbb{Z}$, $a_n > 0$ and $\sum_{n\in\mathbb{Z}} a_n = 1$. Starting with a point $x_0 \in S^1$ let $x_n = R_\alpha^n(x_0)$ for $n \in \mathbb{Z}$. Split open S^1 at each x_n and insert an interval I_n of length a_n. This yields a new circle $\hat{S}$ of perimeter $= 2$

Define $\hat{f} : \hat{S} \circlearrowleft$ by sending $I_n \to I_{n+1}$ linearly and $\hat{f} = f$ on $\hat{S} - \cup I_n$. Then $\underline{X} = \hat{S} - \cup \mathring{I}_n$ where $\mathring{I}_n$ = Interior of I_n.



(1)

Theorem: Assume $f: S^1 \circlearrowleft$ is an orientation preserving homeomorphism with $\rho(f) = \alpha \notin \mathbb{Q}$.

Recall that f has a unique recurrent set (minimal) $\Sigma = \omega(x)$ for all $x \in S^1$

(a) $\exists$ a semi conjugacy

$$\begin{array}{ccc} \Sigma & \xrightarrow{f|\Sigma} & \Sigma \\ \beta \downarrow & & \downarrow \beta \\ S^1 & \xrightarrow{R_\alpha} & S^1 \end{array}$$

where $R_\alpha(\theta) = \alpha + \theta$.

(b) Thus if f is transitive and so $\Sigma = S^1$,

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ \beta \downarrow & & \downarrow \beta \\ S^1 & \xrightarrow{R_\alpha} & S^1 \end{array}$$

with \beta a homeomorphism

(c) When f is not transitive, β extends to a monotone map (preimages are intervals) $\hat{\beta}: S^1 \to S^1$

with

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ \hat{\beta} \downarrow & & \downarrow \hat{\beta} \\ S^1 & \xrightarrow{R_\alpha} & S^1 \end{array}$$

(2)

The main lemma says that when $\rho(f) = \alpha \notin \mathbb{Q}$ then the orbits of f are ordered around the circle in the same manner as those of $R_\alpha(\theta) = \theta + \alpha$.

It is formulated in the universal cover $\mathbb{R}$.

Lemma: Assume $\tilde{f}$ is a lift of f with $\rho(\tilde{f}) = \alpha \notin \mathbb{Q}$

$\forall \tilde{x} \in \mathbb{R}, \forall m_1, m_2, n_1, n_2 \in \mathbb{Z}$

$$\tilde{f}^{n_1}(\tilde{x}) + m_1 < \tilde{f}^{n_2}(\tilde{x}) + m_2 \iff n_1\alpha + m_1 < n_2\alpha + m_2$$

[note that this is $\tilde{R}_\alpha^{n_1}(0) + m_1 < \tilde{R}_\alpha^{n_2}(0) + m_2$]

Proof: Assume $\tilde{f}^{n_1}(\tilde{x}) + m_1 < \tilde{f}^{n_2}(\tilde{x}) + m_2$

Taking $\tilde{f}^{-n_2}$ of both sides and using the fact that f is degree 1,

$$\tilde{f}^{n_1-n_2}(\tilde{x}) < \tilde{x} + m_2 - m_1$$

Now this inequality must hold for all $\tilde{x}$, or else by the IVT there is an $\tilde{x}$ with equality and thus f has a periodic orbit and so $\rho(\tilde{f}) \in \mathbb{Q}$, a contradiction.

Thus, in particular, for $\tilde{x} = 0$ we have

$$\tilde{f}^{n_1-n_2}(0) < m_2 - m_1$$

Using the fact that $\tilde{f}$ is order preserving and degree one

$$\tilde{f}^{n_1-n_2}(\tilde{f}^{n_1-n_2}(0)) < \tilde{f}^{n_1-n_2}(m_2-m_1)$$

$$= \tilde{f}^{n_1-n_2}(0) + m_2 - m_1 < (m_2-m_1) + (m_2-m_1)$$

Continuing (or by induction)

$$\tilde{f}^{k(n_1-n_2)}(0) < k(m_2-m_1)$$

Now assume $n_1 - n_2 \geq 0$ (the other case is similar)

Then $$\frac{\tilde{f}^{k(n_1-n_2)}(0)}{k(n_1-n_2)} < \frac{m_2-m_1}{n_1-n_2}$$ we get using $k(n_1-n_2)$ as a subsequence

Since we know $\rho(\tilde{f})$ exists

$$\alpha = \rho(\tilde{f}) = \lim_{k\to\infty} \frac{\tilde{f}^{k(n_1-n_2)}(0)}{k(n_1-n_2)} \leq \frac{m_2-m_1}{n_1-n_2}$$

but $\alpha \notin \mathbb{Q}$, so equality cannot hold, so

$\alpha(n_1-n_2) < m_2 - m_1$ or $\alpha n_1 + m_1 < \alpha n_2 + m_2$

Finishing the proof of $(\Leftarrow)$

Now the contrapositive of what we just proved is also true for free and it is

$n_1\alpha + m_1 \geq n_2\alpha + m_2 \Rightarrow \tilde{f}^{n_1}(\tilde{x}) + m_1 \geq \tilde{f}^{n_2}(\tilde{x}) + m_2$

and equality cannot hold since $\alpha = \rho(\tilde{f}) \notin \mathbb{Q}$, which proves $(\Rightarrow)$ $\blacksquare$

Proof of Theorem:

- Pick $x_0 \in X$ and $\tilde{x}_0 \in \mathbb{R}$ with $\pi(\tilde{x}_0) = x_0$

- Let $A = \{\tilde{f}^n(\tilde{x}_0) + m : n, m \in \mathbb{Z}\}$ and note that $A = \pi^{-1}(O(x_0, f))$

- Let $B = \{n\alpha + m : n, m \in \mathbb{Z}\} = \{\tilde{R}_\alpha^n(0) + m : n, m \in \mathbb{Z}\}$ and note that $B = \pi^{-1}(O(0, R_\alpha))$ and since R_α is minimal, B is dense in $\mathbb{R}$

- Define $H : A \to B$

$$H(\tilde{f}^n(\tilde{x}_0) + m) = n\alpha + m$$

- By the Lemma, H is bijective and order preserving

We claim that $\tilde{H}$ extends to a continuous map $\hat{H} : \bar{A} \to \bar{B} = \mathbb{R}$. The explicit formula is

$$\hat{H}(y) = \sup\{n\alpha + m : \tilde{f}^n(\tilde{x}_0) + m > y\}$$

Now note that since $\mathbb{R} - B$ can't contain an interval

$$\hat{H}(y) = \inf\{n\alpha + m : \tilde{f}^n(\tilde{x}_0) + m < y\}.$$

which implies the continuity of $\hat{H}$.

Now

- $$\hat{H}(y+1) = \sup\{n\alpha + m : \tilde{f}^n(\tilde{x}_0) + m < y+1\}$$
$$= \sup\{n\alpha + m : \tilde{f}^n(\tilde{x}_0) + m - 1 < y\} = \hat{H}(y) + 1$$

- $$\hat{H}(\tilde{f}(y)) = \sup\{n\alpha + m : \tilde{f}^n(\tilde{x}_0) + m < \tilde{f}(y)\}$$
$$= \sup\{n\alpha + m : \tilde{f}^{n-1}(\tilde{x}_0) + m < y\}$$
$$= \alpha + \tilde{f}(y)$$

$$\pi(\bar{A}) = \bar{X}$$

Thus $\hat{H}$ descends to a map β with

$$\begin{array}{ccc} \bar{X} & \xrightarrow{f|\bar{X}} & \bar{X} \\ \beta \downarrow & & \downarrow \beta \\ S^1 & \xrightarrow{R_\alpha} & S^1 \end{array}$$

Finishing (a). (b) follows directly from (a)

For (c), define $\bar{H} : \mathbb{R} \to \mathbb{R}$ via

$$\bar{H}(y) = \sup\{n\alpha + m;\ \tilde{f}^n(x) + m < y\}$$

which obviously extends $\hat{H}$ and is constant on the gaps of $\bar{A}$ (recall $\bar{X}$ is a cantor set)

and $\bar{H}(y+1) = \hat{H}(y) + 1$ and $\bar{H}(\tilde{f}(y)) = \alpha + \tilde{f}(y)$

which descends to $\hat{\beta}$.

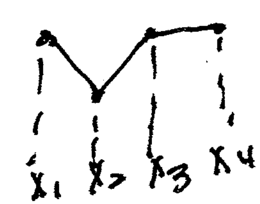
- The next result is due to Denjoy 1932. It says if f is "regular" enough then f is always transitive and thus topologically conjugate to rigid rotation

- We need some definitions from Real Analysis:

The total variation of a function $g: S^1 \to \mathbb{R}$ is $\text{Var}(g) = \sup \sum_{k=1}^{n} |g(x_k) - g(x_{k+1})|$

where the sup is over all partitions $0 \leq x_1 < x_2 < \ldots < x_n \leq 1$ for all $n \in \mathbb{N}$

(note: we are treating S^1 as $[0,1]/\sim$



- If g is Lipschitz ($\exists L$: $|g(y_1)-g(y_2)| < L|y_1-y_2|$) $\forall g$

then g has $Var(g) < \infty$.

- If $g \in C^1$ i.e. g' exists and is continuous

then $g': S^1 \to \mathbb{R}$ is bounded since g' is continuous and S^1 is compact $\Rightarrow$ by the MVT, g is Lipschitz and thus has $Var(g) < \infty$

- If $Var(g) < \infty$ then g is said to have bounded variation and written $g \in BV$

- The regularity we need on $f: S^1 \circlearrowleft$ for Denjoy's theorem is that the derivative is BV.

- DEF: $f \in C^{1+BV}(S^1)$ if f is differentiable, f' is continuous and $f' \in BV(S^1)$.

<u>Denjoy</u>. Assume $f: S^1 \circlearrowleft$ is a diffeomorphism (both f and f^{-1} are C^1) and $f \in C^{1+BV}(S^1)$ and $\rho(f) = \alpha \notin \mathbb{Q} \Rightarrow f$ is topologically conjugate to (S^1, R_α)

<u>Lemma</u>: Let $J \in S^1$ be a nontrivial interval with $\mathring{J}, f(\mathring{J}), \ldots, f^{n-1}(\mathring{J})$ disjoint for some n. Let $g = \log(f')$ and for $x, y \in \bar{J}$ we have

$$\mathrm{Var}(g) \geq \left| \log[(f^n)'(y)] - \log[(f^n)'(x)] \right|$$

$$= \left| \log \frac{(f^n)'(y)}{(f^n)'(x)} \right|$$

Proof: Since $x, y \in J$ and $\mathring{J}, f(\mathring{J}), \dots, f^{n-1}(\mathring{J})$ are pairwise disjoint, assuming $y < x$, then $[y,x], f([y,x]), \dots, f^{n-1}([y,x])$ is contained in a partition of S^1, so by the def. of $\overline{\mathrm{Var}}(g)$

$$\overline{\mathrm{Var}}(g) \geq \sum_{k=0}^{n-1} \left| g(f^k(y)) - g(f^k(x)) \right|$$

triangle $\neq$

$$\geq \left| \sum_{k=0}^{n-1} g(f^k(y)) - g(f^k(x)) \right|$$

def g

$$= \left| \sum_{k=0}^{n-1} \log f'(f^k(y)) - \log f'(f^k(x)) \right|$$

$$= \left| \log \prod_{k=0}^{n-1} f'(f^k(y)) - \log \prod_{k=0}^{n-1} f'(f^k(x)) \right|$$

$$= \left| \log [(f^n)'(y)] - \log [(f^n)'(x)] \right|$$

$$= \left| \log \left((f^n)'(y) / (f^n)'(x) \right) \right|$$

Proof of Denjoy Theorem next lecture