Denjoy's Theorem

- It says that if an orientation preserving diffeomorphism has sufficient regularity and $\rho(f) = \alpha \notin \mathbb{Q}$ then f is transitive and so by Poincaré's Theorem, it is topologically conjugate to rigid rotation by α. Recall the Lemma from last lecture.

- **Lemma**: J is nontrivial interval in S^1 and $\mathring{J}, f(\mathring{J}), \ldots, f^{n-1}(\mathring{J})$ are pairwise disjoint, $g = \log(f')$. For any $x, y \in J$

$$\mathrm{Var}(g) \geq \left| \log\left(\frac{(f^n)'(x)}{(f^n)'(y)} \right) \right|$$

<u>Denjoy's Theorem</u>: Assum f is a C^1, orientation preserving diffeomorphism of S^1 and $\rho(f) = \alpha \notin \mathbb{Q}$ and $f \in C^{1+BV}(S^1) \Rightarrow f$ is transitive and thus by Poincaré, topologically conjugate to R_α.

<u>Proof</u> By contradiction. So assume f is not transitive and and so f has a Cantor set minimal set $\underline{X}$ and $S^1 - \overline{X} = \bigcup I_n$ with I_n an open <u>interval</u>. Pick $I = I_0$ and note that $\{f^n(I)\}_{n\in\mathbb{Z}}$ are pairwise disjoint or else f has a periodic point in contradiction to $\rho(f) \notin \mathbb{Q}$. Let $\ell(J)$ be the length of the interval J. We know that

$$\sum_{n\in\mathbb{Z}} \ell(f^n(I)) \leq 1$$

and that $\ell(f^n(I)) =$

$$\int_I (f^n)'(t)\, dt.$$

<u>Claim</u>: Fix $x \in S^1$ one of these (or both) hold

(a) There are infinitely many $n > 0$ so that the intervals $(x, f^{-n}(x)), (f(x), f^{1-n}(x)), \ldots, (f^n(x), x)$ are disjoint.

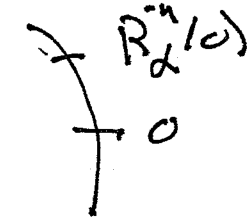
(b) There are infinitely many $n > 0$ so that the intervals $(f^{-n}(x), x), (f^{1-n}(x), f(x)), \ldots, (x, f^n(x))$ are disjoint

<u>Claim Proof</u>

- Since we know that orbits of f are ordered in the same manner as those of R_α, it suffices to prove the result for $f = R_\alpha$ and $x = 0$

- Since $o(0, R_\alpha^{-1})$ is dense, 0 has infinitely many closest returns to itself, i.e. n so that $d(0, R_\alpha^{-n}(0)) < d(0, R_\alpha^{-k}(0))$ for $0 < k < n$

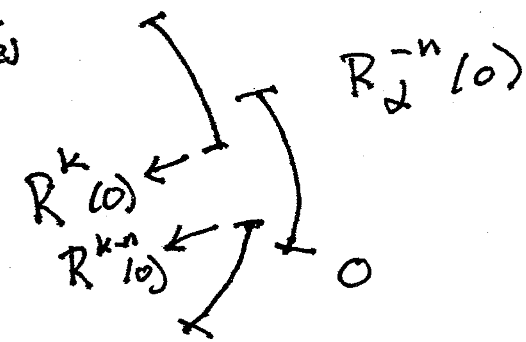
Pick one such n and assume $R_\alpha^{-n}(0) > 0$



Let $J = (0, R_\alpha^{-n}(0))$. We need to show that $J, f(J), \ldots, f^n(J)$ are pairwise disjoint.

If not, for some i, j $f^i(J) \cap f^j(J) \neq 0$ and if $j > i$, $f^{j-i}(J) \cap J \neq \phi$, let $k = j - i$

The two possibilities are



Thus it suffices to show that

$\forall\ 0 < |k| < n, \qquad R^k(0) \notin [0, R_\alpha^{-n}(0)]$

Now by the choice of n as a closest return of R^{-1}, we know $R^{-k}(0) \notin [0, R^{-n}(0)]$ for $0 < k < n$. Now assume some $R^{k}(0) \in [0, R^{-n}(0)]$ and so $R^{-n+k}(0) \in [R^{-n}(0), R^{-2n}(0)]$ and

so $d(R^{-(n-k)}(0), R^{-n}(0)) < d(R^{-n}(0), R^{-2n}(0))$

$= d(0, R^{-n}(0))$ and $0 < n-k < n$, a contradiction in the case $R^{-n}(0) > 0$.

This completes the proof in the case $R^{-n}(0) > 0$. The case $R^{-n}(0) < 0$ is similar. Since one of these cases must happen infinitely often, we are done ∎

Now BACK TO the main proof. We assume that part (a) of the lemma hold, part (b) is similar let $J = (x, f^{-n}(x))$

and we know for infinitely many $n > 0$ by the claim

$J, f(J), f^2(J), \ldots, f^n(J)$ are pairwise disjoint

Pick one such n, and let $y = f^{-n}(x)$ and

so $(x, y) \in J$ and note by the chain rule (or inverse derivative)

$$(f^n)'(y) = (f^{+n})'(f^{-n}(x)) = \frac{1}{(f^{-n})'(x)}$$

[differentiate $f^n \circ f^{-n}(x) = x$]

Thus the Lemma says with $g = \log f'$

$$\operatorname{Var}(g) \geq \left| \log \frac{(f^n)'(x)}{(f^n)'(y)} \right| = \left| \log\left((f^n)'(x) \cdot (f^{-n})'(x) \right) \right|$$

and so $(f^n)'(x) \cdot (f^{-n})'(x) \geq \exp(-\operatorname{Var}(g))$ (*)

Thus

$$
\begin{aligned}
\ell(f^n(I)) + \ell(f^{-n}(I)) &= \int_I (f^n)'(x) + (f^{-n})'(x) \, dx \\
&\geq \int_I \sqrt{(f^n)'(x)\,(f^{-n})'(x)} \, dx \\
&\geq \int_I \left(\exp(-\mathrm{Var}(g))\right)^{1/2} dx \\
&= \left(\exp(-\mathrm{Var}(g))\right)^{1/2} \cdot \ell(I).
\end{aligned}
$$

By assumption $f' \in BV$ $\Rightarrow$ $g = \log f' \in BV$ $\Rightarrow$ By assumption $\mathrm{Var}(g) < \infty$, so $C = \exp(-\mathrm{Var}(g))^{1/2} > 0$

Thus $\ell(f^n(I)) + \ell(f^{-n}(I)) \geq C\,\ell(I)$ for infinitely n, contradiction to $\sum_{n \in \mathbb{Z}} \ell(f^n(I)) < \infty$ ∎

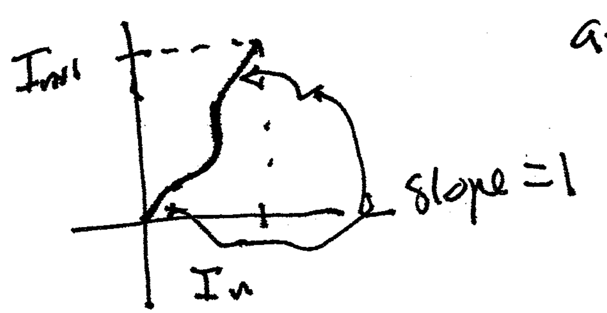
<u>Denjoy's example</u>: $\forall \alpha \notin \mathbb{Q}$, $\exists$ orientation preserving diffeomorphism f with $\rho(f)=\alpha$ and f is C^1 and has a Denjoy minimal set and thus is not topologically conjugate to R_α.

<u>Idea</u> Proceed as in the topolgical construction but chose a_n, I_n and the maps $I_n \to I_{n+1}$ carefully

(a) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ (fails the ratio test)

(b) $\sum\limits_{n\in\mathbb{Z}} a_n = 1$

(c) $f' = 1$ on the endpoints of each I_n and on all of $X = S^1 - \bigcup \mathring{I}_n$



<u>Question</u>: If f is a very smooth diffeomorphism may we find a smooth h with. $hfh^{-1} = R_\alpha$ with $\alpha = \rho(f)$.

- Lots of deep and subtle results. Here's one not to technical

- (Hermann, Yoccuz) If f is a C^∞-diffeomorphism and $\rho(f) = \alpha$ satisfies a Diophantine condition (so it's "a long way from rational) the $\exists$ C^∞-diffeomorph h with $hfh^{-1} = R_\alpha$

- This implies f preserves a nice measure βdm with β a C^∞-density.