- Recall $h: X \circlearrowleft$ compact for homeomorphism h is **minimal** if $Z \subseteq X$ cpt and $h(Z) = Z$ implies $Z = X$ or $Z = \emptyset$.
- So minimal means dynamically ~~dec~~ indecomposable i.e. no nontrivial sub-invariant sets that are compact
- There are many alternative characteristics of minimal
- First, the simplest compact invariant sets are orbit closures $\overline{O(x,h)}$ (over bar is closure.)

Proof: cpt since closed for invariant if $y \in \overline{O(x,h)}$ ~~certa~~ certainly $h(y)$ and $h^{-1}(y) \in \overline{O(x,h)}$

and if $h^{n_i}(x) \to y$ Then $h^{n_i+1}(x) \to h(y) \in \overline{O(x,h)}$

and $h^{n_i-1}(x) \to h^{-1}(y) \in \overline{O(x,h)}$ so $h(\overline{O(x,h)}) \subseteq \overline{O(x,h)}$ and $h^{-1}(\overline{O(x,h)}) \subseteq \overline{O(x,h)}$

- ~~Recall~~ Alternative proof: Show that $\overline{o(x,h)} = o(x,h) \cup \alpha(x) \cup \omega(x)$ and each piece is h-invariant

- Recall $Z \subseteq X$ is dense if $\overline{Z} = X$ alternatively, every $B_\varepsilon(x)$ for $\varepsilon > 0$, $x \in X$ contains a point of Z.

- **Theorem:** (X, h) is minimal $\iff$ $\forall x \in X$, $\overline{o(x,h)} = X$

  i.e. every orbit is dense

- Proof: we prove the contrapositive

  (X, h) is not minimal $\iff$ $\exists x \in X$ with $\overline{o(x,h)}$ properly contained in X.

Assume $(\overline{X}, h)$ is not minimal so there exists $Z \subsetneq \overline{X}$ compact, invariant. Thus for any $z \in Z$, $o(z,h) \subseteq Z$ since Z is h invariant and $\overline{o(z,h)} \subseteq \overline{Z} = Z$ since Z is closed so $\overline{o(z,h)} \subsetneq \overline{X}$

Now conversely, assume $\exists x$ with $\overline{o(x,h)} \subsetneq \overline{X}$ but then as noted above $\overline{o(x,h)}$ is a compact, invariant, nonempty set and it is properly contained in $\overline{X}$ so $(\overline{X}, h)$ is not minimal. ∎

Theorem: If $\omega \notin \mathbb{Q}$ then $R_\omega : S^1 \supseteq$ defined by $\theta \to \theta + \omega \mod 1$ is minimal.

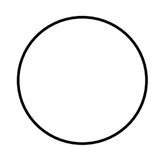
For the proof we need some new notions

(1) $Z \subseteq X$ is called ε-dense if for every epsilon ball $B_\varepsilon(x)$, $Z \cap B_\varepsilon(x) \neq \emptyset$. By the characterization above, Z is dense $\Leftrightarrow$ it is ε-dense for all $\varepsilon > 0$.

(2) $h : (X, d) \supsetneq$ is an isometry if it preserves the metric i.e. $d(h(x), h(y)) = d(x, y)$, $\forall x, y \in X$

Note that this implies that h^n is an isometry $\forall n \in \mathbb{Z}$

(3) R_w is an isometry of the usual metric on S^1



Now for the proof. First note that $R_\omega^n(x) \neq x$

for any x ~~be~~ and $n \neq 0$ cause if $R_\omega^n(x) = x \mod 1$

$x + n\omega = x + p$ some $p \in \mathbb{Z}$ so $\omega = \frac{p}{n} \in \mathbb{Q}$, contradiction

Now fix x and consider the set $\{x, R_\omega(x), \ldots, R_\omega^{m-1}(x)\} \subseteq S^1$

These are m distinct points in S^1, Thus there must

be some pair of them within $1/m$ of each other.

or $\exists$ $a > b$, $a, b < m$ with $d(R_\omega^a(x), R_\omega^b(x)) < 1/m$

but R_ω^{-b} is an isometry so $d(R_\omega^{a-b}(x), x) < 1/m$

using the fact that R_ω^{a-b} is an isometry,

$d(R_\omega^{(a-b)n}(x), R_\omega^{(a-b)(n-1)}[x]) < 1/m$ for all n.

Thus $\mathcal{O}(x, R_\omega^{a-b})$ is $1/m$ dense in S^1

But $o(x, R_\omega^{a-b}) \subseteq o(x, R_\omega)$ and thus $o(x, R_\omega)$ is $1/m$-dense for all m and so $o(x, R_\omega)$ is dense in S^1 $\blacksquare$

- There is much more to say about minimality, but we save that until we have more examples.

- A ~~weak~~ form of indecomposibility that is weaker than minimality is transitivity

DEF: $h: X \circlearrowleft$ is transitive if there exists a point x with a dense orbit or $\overline{o(x, h)} = X$

Remark: Minimal $\Rightarrow$ transitive but not the converse. (eg (Σ_2^+, σ) defined next lecture)

For the next theorem we need to know about the Baire Theorem. First a definition, A set $Z \subseteq X$ is called G_δ if $Z = \bigcap_{n=1}^{\infty} U_n$ with each U_n open in X.

Exercise: Show the irrationals are G_δ in $[0,1]$

The version of the Baire Category we need is

Theorem: If X is a complete metric space (Cauchy sequences converge) and U_i is open dense for $i \in \mathbb{N} \Rightarrow \bigcap_{i=0}^{\infty} U_i$ is dense (also G_δ by def)

Remark: Compact metric spaces are complete.

8B

We also need some basic topology

- A collection $\{U_\lambda\}_{\lambda\in\Lambda}$ is a base for the topology if any open U can be written $U = \bigcup U_\alpha$ for some $U_\alpha \subseteq \{U_\lambda\}$.

- A topological space is second countable if it has a countable base $\{U_n\}_{n\in\mathbb{N}}$

- A compact metric space is second countable

Proof: Pick $n\in\mathbb{N}$, then $\{B_{\frac{1}{n}}(x) : x\in X\}$ is a cover of X, so by compactness it has a finite subcover, say $U_{n,1}, \dots, U_{n,m(n)}$

Then $\bigcup_{n\in\mathbb{N}} \{U_{n,1}, \dots, U_{n,m(n)}\}$ is a countable bas.

<u>Theorem</u>: TFAE for $h: X \circlearrowleft$ a homeomorphism of a compact metric space

(a) h is topologically transitive

(b) If $Z \subseteq X$ is compact, invariant then either $Z = X$ or Z is nowhere dense (means $\bar{Z}$ has no interior)

$\bar{Z} = Z$ since compact

(c) If u, v are nonempty open ... then $\exists\, n \in \mathbb{Z}$ with $h^n(u) \cap V \neq \emptyset$

(d) $\{x \in X : \overline{o(x,h)} = X\}$ is dense G_δ

Interpretation: (b) Indecomposability

(c) "mixing"

(d) once there is a single dense orbit there are lots of them

dense, G_δ = topologically big.

Proof:

(a) $\Rightarrow$ (b) By hypothesis there is an x_0 with $\overline{o(x_0,h)} = X$ and a compact Z with $h(Z) = Z$.
We show that if Z is not nowhere dense i.e. $\exists$ open, nonempty U with $U \subseteq \overline{Z} = Z$ then $Z = X$, proving (b). To prove this since $o(x_0,h)$ is dense and U is open, $\exists\, n$ with $h^n(x_0) \in U \subseteq Z$. But $h(Z) = Z$ so
$$o(x_0,h) \subseteq Z \text{ and thus } X = \overline{o(x_0,h)} \subseteq \overline{Z} = Z$$
so $Z = X$.

(b) $\Rightarrow$ (c) First notice condition (b) is equivalent to this: If $U \neq \phi$, is open and $h(U) = U$ then either $U = X$ or U is dense in X.

Now to prove (c), given some $\mathcal{U}$ form

$$W = \bigcup_{n \in \mathbb{Z}} h^n(\mathcal{U}) .$$

This is open and ~~s~~ $h(W) = W$ as is easy to check. Thus by what we showed (b) is equivalent to, W is dense in X. Thus for any open V, $W \cap V \neq \emptyset$ and so $\exists\, n$ with $h^n(\mathcal{U}) \cap V \neq \emptyset$.

(c) $\Rightarrow$ (d) Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a countable basis for X. For each j, let $V_j = \bigcup_{n \in \mathbb{Z}} h^{-n} \mathcal{U}_j$.

Note that we also have $V_j = \{x : \exists\, n \text{ with } h^n(x) \in \mathcal{U}_j\}$.

~~Thus since~~ Since $h^n(x) \in \mathcal{U}_j \iff x \in h^{-n}(\mathcal{U}_j)$

Now (c) says that given an open V $\exists n$ so that $h^n(U_j) \cap V \neq \phi$, Thus $V_j = \bigcup h^{-n} U_j$. also satisfies $V_j \cap V \neq \phi$. Thus V_j intersects every open set V and so V_j is dense.

Thus by the Baire Category Theorem

$$\bigcap_{j=1}^{\infty} V_j \text{ is dense, } G_\delta$$

but recall $V_j = \{x : \exists n \text{ with } h^n(x) \in U_j\}$

and so $\bigcap V_j = \{x : \forall U_j \ \exists n \text{ with } h^n(x) \in U_j\}$

i.e. $\bigcap V_j$ is precisely $\{x : \overline{O(x,h)} = X\}$

(d) $\Rightarrow$ (a) is trivial.