Attractors

- Informally, a set $\Lambda \subseteq X$ is an attractor for $f: X \to X$ if all nearby points converge to it and thus it is physically visible

- We mainly work with discrete dynamical systems making a few remarks about flows

- Here's an example: Let T be the solid torus $D^2 \times S^1$ ($D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$) living in $\mathbb{R}^3$.

- Stretch it to twice its length and wrap it around twice and put it back into itself

- Call this map $f: T \to T$ so f is a homeomorphism onto its image but is not onto.
- Keep iterating so we get $T \supseteq f(T) \supseteq \dots \supseteq f^n(T)$
- The attractor (precisely defined below) is

$$\Lambda = \bigcap_{n=0}^{\infty} f^n(T)$$

- We shall prove many features of Λ
  (1) Λ is connected but not locally connected
  (2) Λ is locally the product of a Cantor set and an interval
  (3) $f|\Lambda$ has dense periodic points, is transitive and has sensitive dependence on intial conditions
  (4) $f|\Lambda$ can be modeled by an inverse limit. (to be defined).

- By standard methods f can be extended to $f: \mathbb{R}^3 \to \mathbb{R}^3$ so that $\forall x \in \mathbb{R}^3$, $\exists\, n$ with $f^n(x) \in T$. But we just focus on f acting on T.

- Before analyzing the example (called the solenoid) we need some generalities.

- Assume $f: X \to X$ is a homeomorphism of a metric space.

- A set N is called a <u>trapping region</u> if
  (a) $\overline{N}$ is compact
  (b) $f(\overline{N}) \subseteq \mathring{N}$

  $\left\{ \begin{array}{l} \mathring{A} = \text{Interior}(A) \\ \overline{A} = \text{closure}(A) \end{array} \right.$

  and the associated attractor is

$$\Lambda = \bigcap_{n=0}^{\infty} f^n(\overline{N})$$

- Note that $\Lambda \neq \emptyset$ since $\overline{N}$ is compact

<u>Exercise</u>: Λ is completely invariant, $f(\Lambda) = \Lambda$.

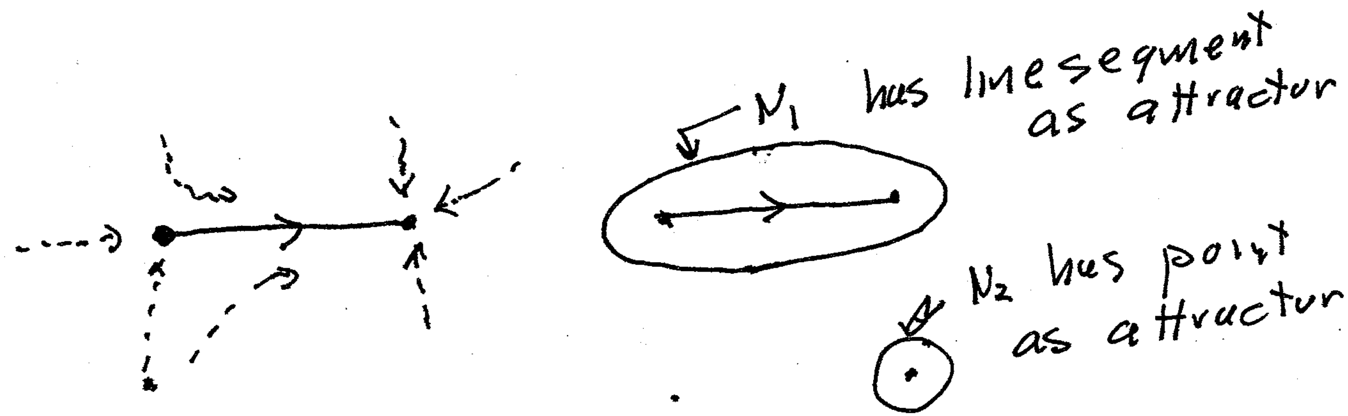
- A set Λ is called an <u>attractor</u> or <u>attracting set</u> if $\exists$ a trapping region N with $\Lambda = \bigcap_{n=0}^{\infty} f^n(\bar{N})$

- <u>FACT</u>: If Λ is an attractor with trapping region $N \Rightarrow \forall x \in N$, $\omega(x) \subseteq \Lambda$ so Λ "contains the future" of every point in N

- A set Λ is called a repeller or repelling set if it is an attractor for f^{-1}.

- If Λ is an attractor with trapping region $N \Rightarrow B = \bigcup_{n=0}^{\infty} f^{-n}(N)$ is the basin of attraction of Λ. (This is independent of the choice of trapping region)

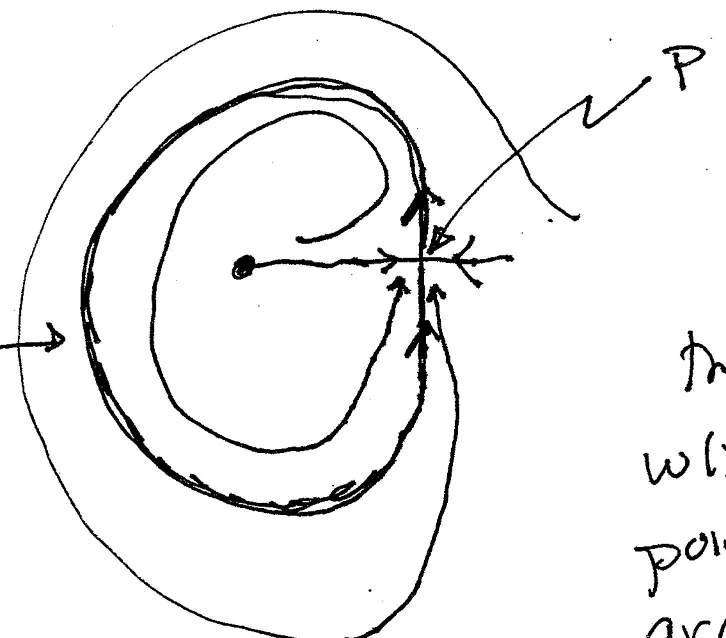
- A problem with the definition of attractor is that there can be attractors inside attractors.

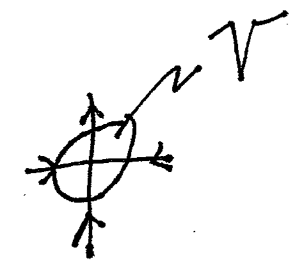


- For this reason some authors require an attractor the be indecomposible in some fashion, e.g. transitive. Other others require sensitive dependence or positive Lyapunov exponents (and then there are "strange attractors" ...)

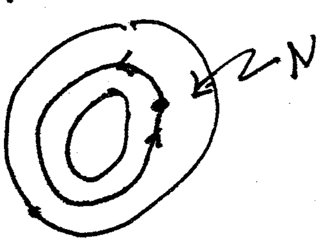
L6+

- A natural question is whether being the "future" of all nearby points makes you an attractor.
- More precisely, say Λ is compact, invariant and $\exists$ an open set $V \supseteq \Lambda$ so that $x \in V \Rightarrow \omega(x) \subseteq \Lambda$. Is Λ an attractor.
- No:





The circle is an attracting set



then $x \in V \Rightarrow \omega(x) = P$ (many points go once around before converging to P)

V is not a trapping region

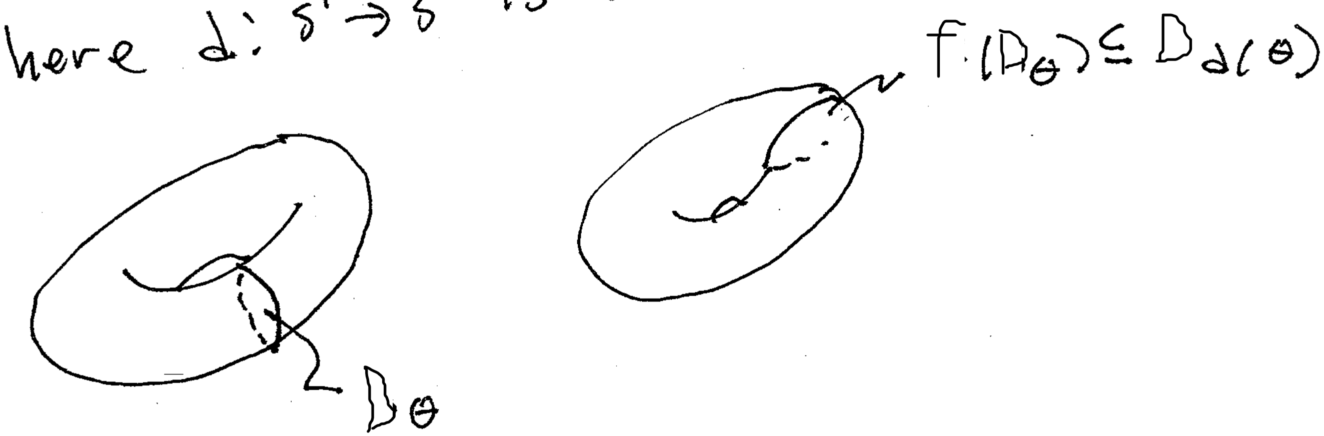
The solenoid in more detail

- We use complex coordinates z on D^2, so $|z| \leq 1$
- The coordinate θ will be used on $S^1 = \mathbb{R}/\mathbb{Z}$
- The solid torus is $T = S^1 \times D^2$
- Define $f : T \to T$ as

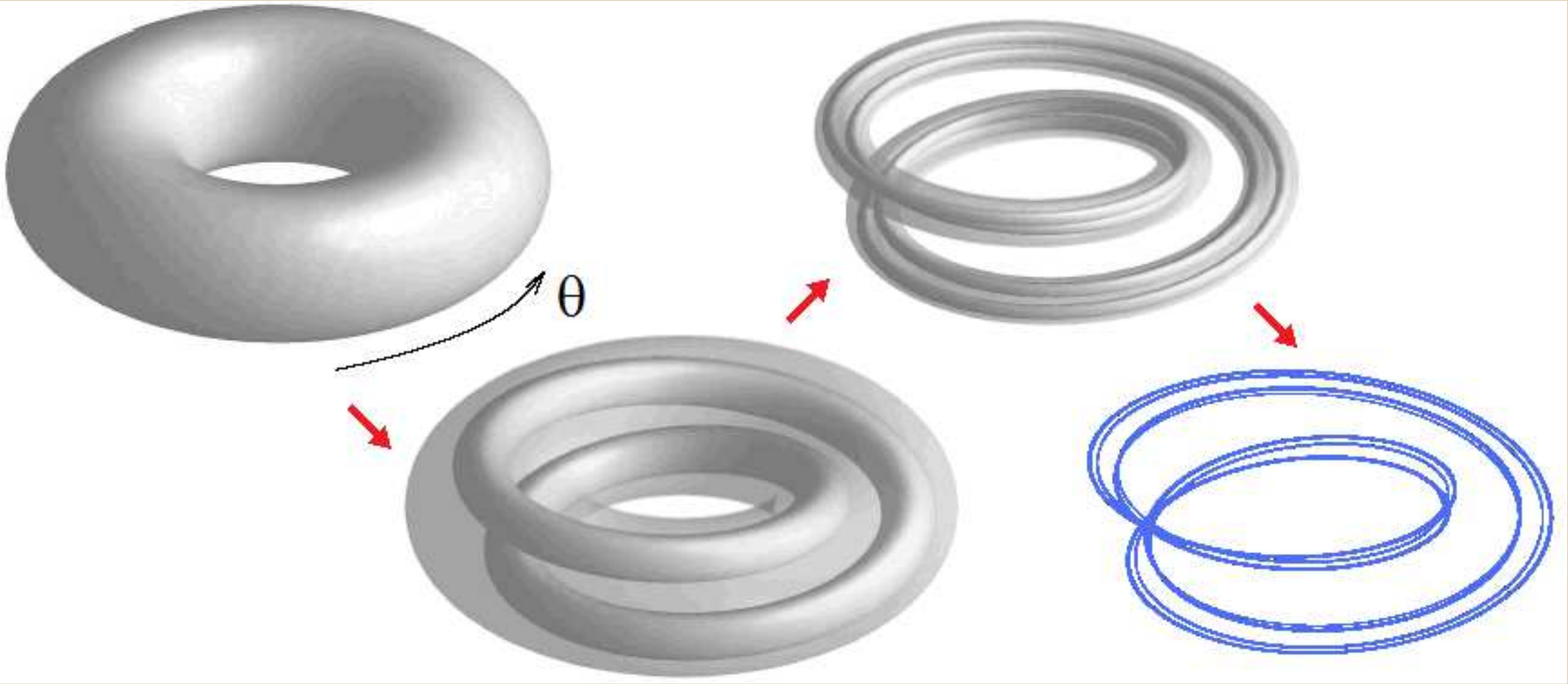
$$f(\theta, z) = \left(2\theta \bmod 1, \frac{1}{10} z + \frac{1}{2} e^{2\pi i \theta}\right)$$

- So if $D_\theta = \{\theta\} \times D^2$, then $f(D_\theta) \subseteq D_{d(\theta)}$

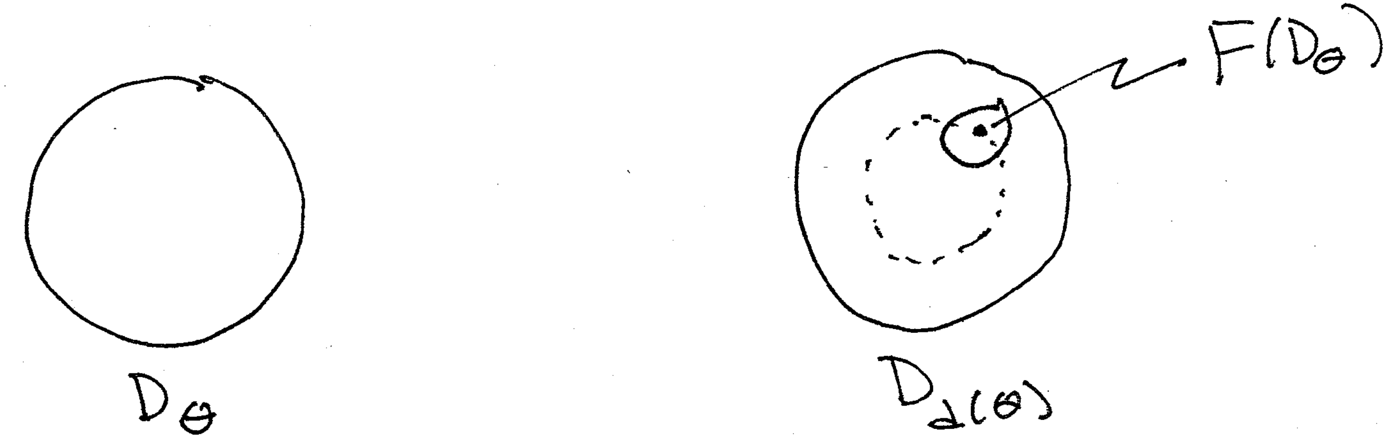
where $d : S^1 \to S^1$ is angle doubling, $d(\theta) = 2\theta \bmod 1$



First example: Williams-Smale solenoid



- What does the image $F(D_\theta)$ look like?



The $\frac{1}{10}z$ shrinks D_θ by $\frac{1}{10}$

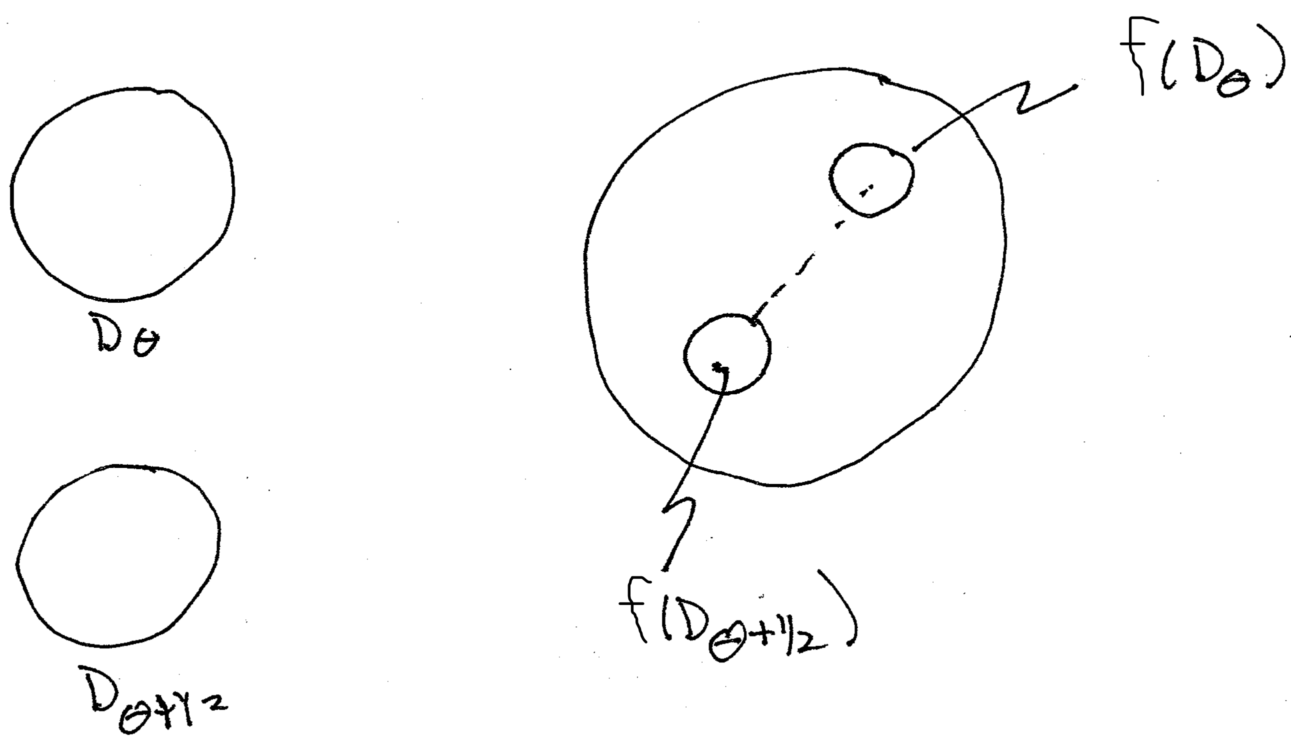
the $+\frac{1}{2}e^{2\pi i\theta}$ puts the image with center

on the circle $|z| = \frac{1}{2}$

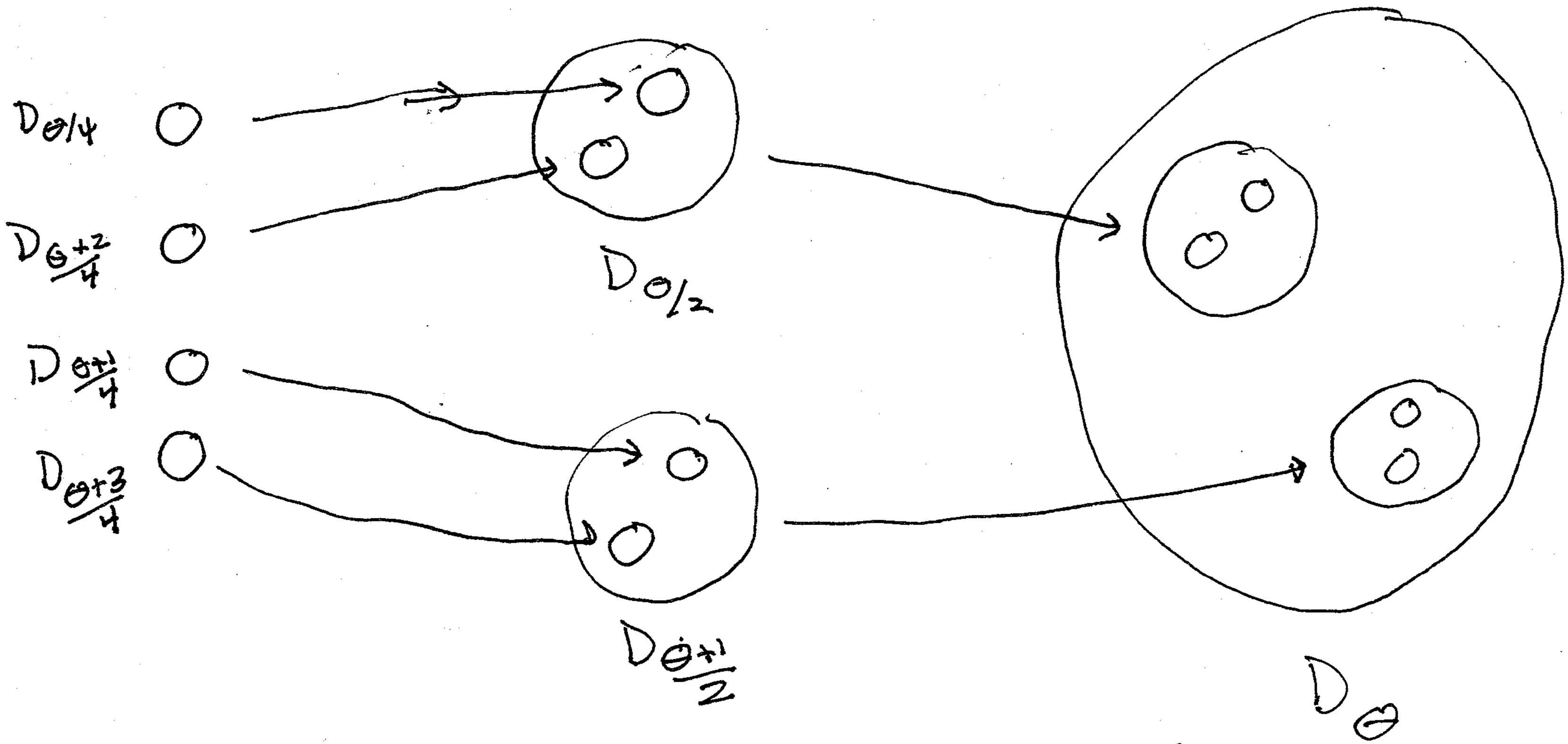
Now notice that $d(\theta+\frac{1}{2}) = 2(\theta+1/2) \bmod 1 = 2\theta = d(\theta)$

So $f(D_{\theta+1/2}) \subseteq D_{d(\theta)}$ also but

in the diametrically opposite side

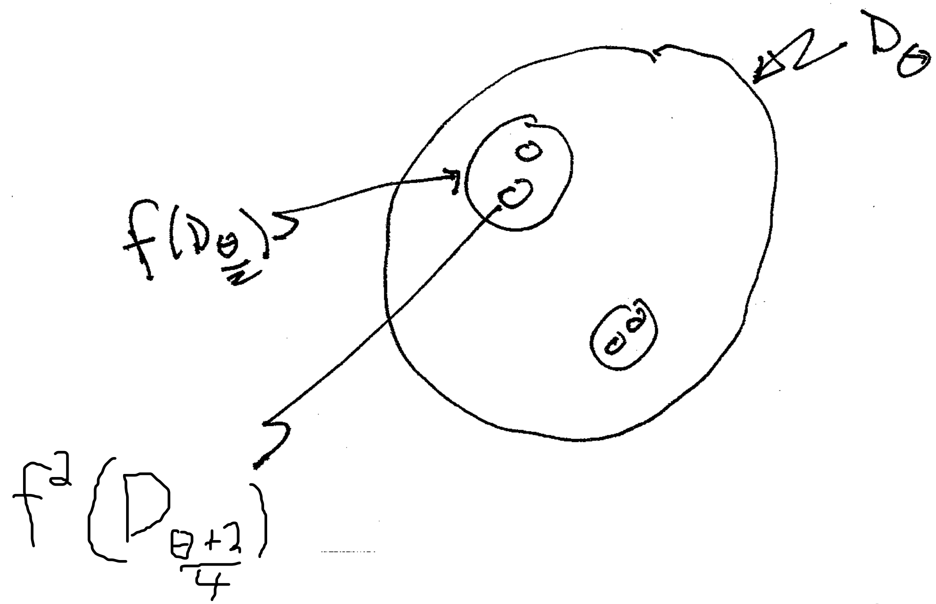


To understand $\Lambda \cap D_\theta$ we back up



- Notice each disk is $\frac{1}{10}$ the radius of previous so, for example, $f^2(D_{\theta/4})$ has radius $\frac{1}{100}$.
- Continuing, we get a construction of a binary Cantor set (perfect, compact, completely disconnected) and homeomorphic to Σ_2^+

But there is a more natural way to code points of $\Lambda \wedge D_\theta$, namely, by the successive disks



so we have started a coding

$\theta, \frac{\theta}{2}, \frac{\theta+2}{4}, \ldots$

- so what are the allowable codes?
- well if the code is $\theta_1, \theta_2, \theta_3, \ldots$

for $f(D_{\theta_2}) \subseteq D_{\theta_1}$ we need $d(\theta_2) = \theta_1$ since f acts in the radial coordinate by $d(\theta) = 2\theta \mod 1$

So the natural coding of $\Lambda \cap D_\theta$ is

$$\{ \theta, \theta_1, \theta_2, \ldots : \theta_n \in S^1 \text{ and } d(\theta_{n+1}) = \theta_n \}$$

which is a piece of the <u>inverse limit</u> (to be studied soon).