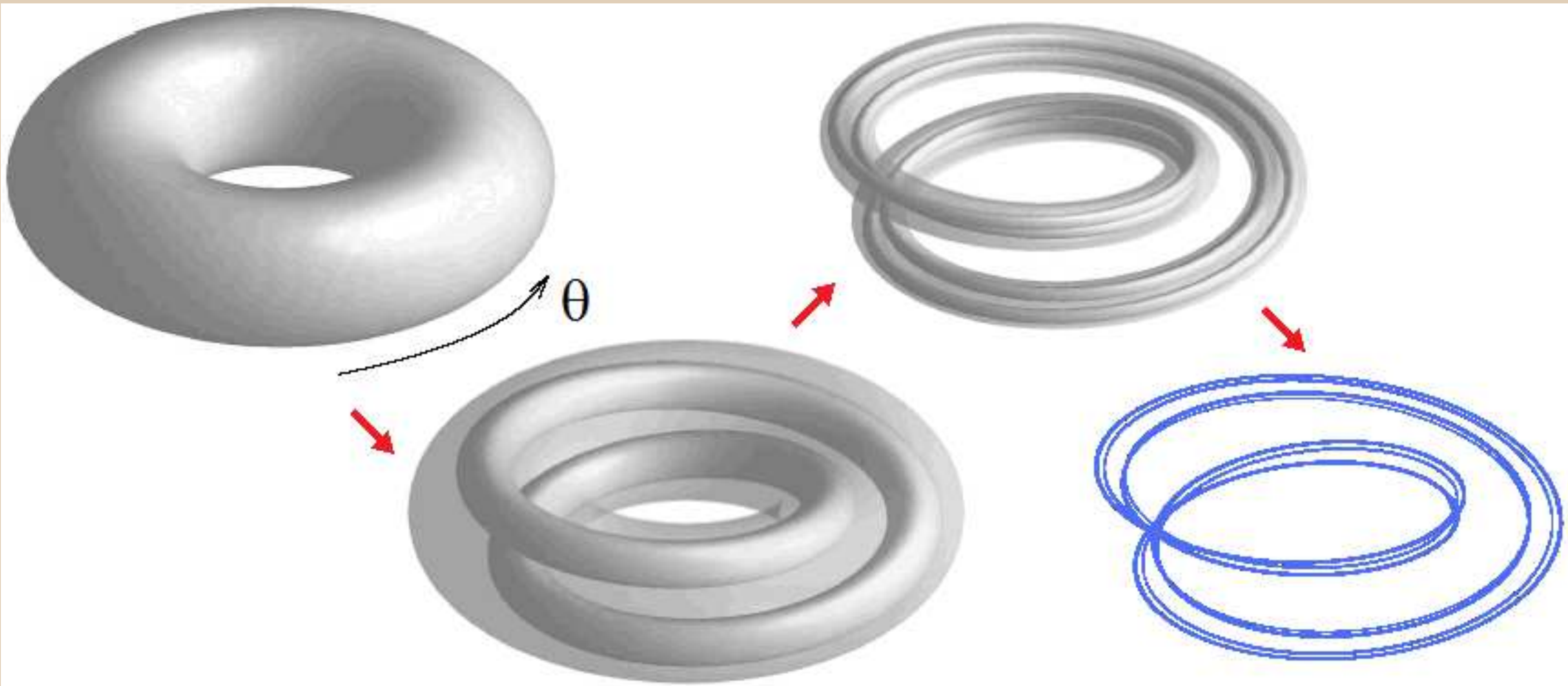
First example: Williams-Smale solenoid



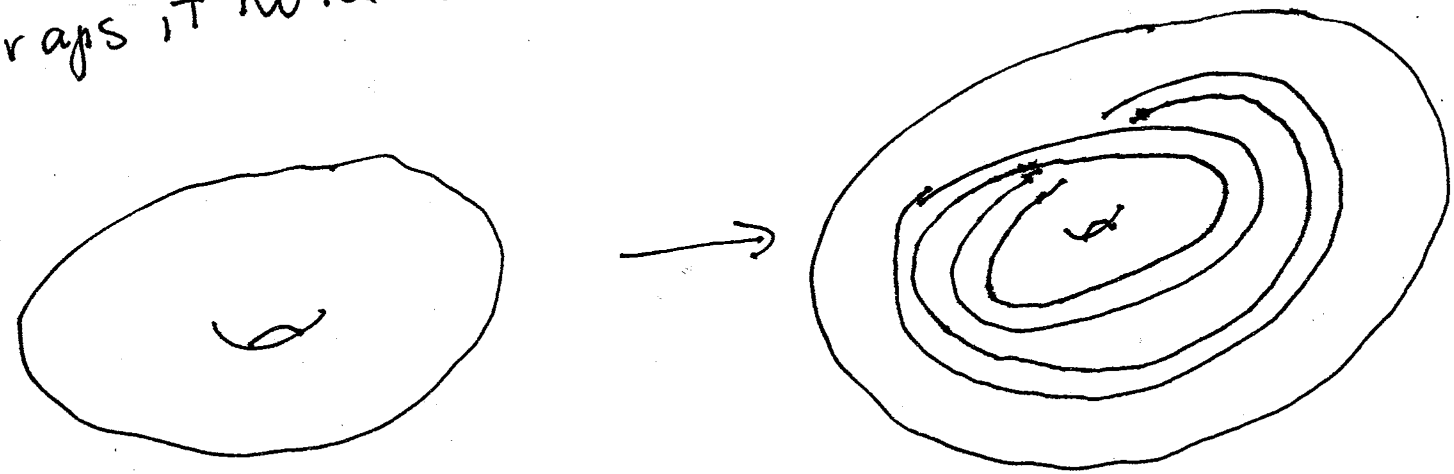
Solenoid Attractor cont

- Solid torus $= T = S^1 \times D^2$ with complex coord $z \in D^2$, $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$
- $f : T \to T$ via

$$f(\theta, z) = \left(d(\theta), \frac{1}{10} z + \frac{1}{2} e^{2\pi i \theta}\right)$$

with $d(\theta) = 2\theta \mod 1$, "angle doubling"

- Takes T and stretches it to twice its length, shrinks its width by $1/10$ then wraps it twice inside T



- $\Lambda = \bigcap_{n=0}^{\infty} f^n(T)$ is an attractor with trapping region T

- $f : T \to T$ can be extended to $\bar{f} : \mathbb{R}^3 \to \mathbb{R}^3$, but we just focus on f mapping T <u>into</u> T.

- <u>Theorem</u>

  (1) Λ is connected and is locally the product of a Cantor set and an interval and so is not locally connected

  (2) $f|_\Lambda$ is transitive, has dense periodic points and sensitive dependence on initial conditions.

- We will look at the proof from two angles
  (a) directly with topology (b) modeling Λ by an inverse limit.

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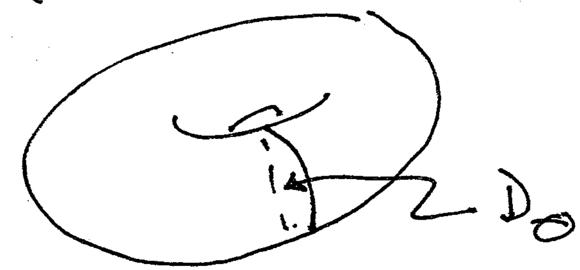
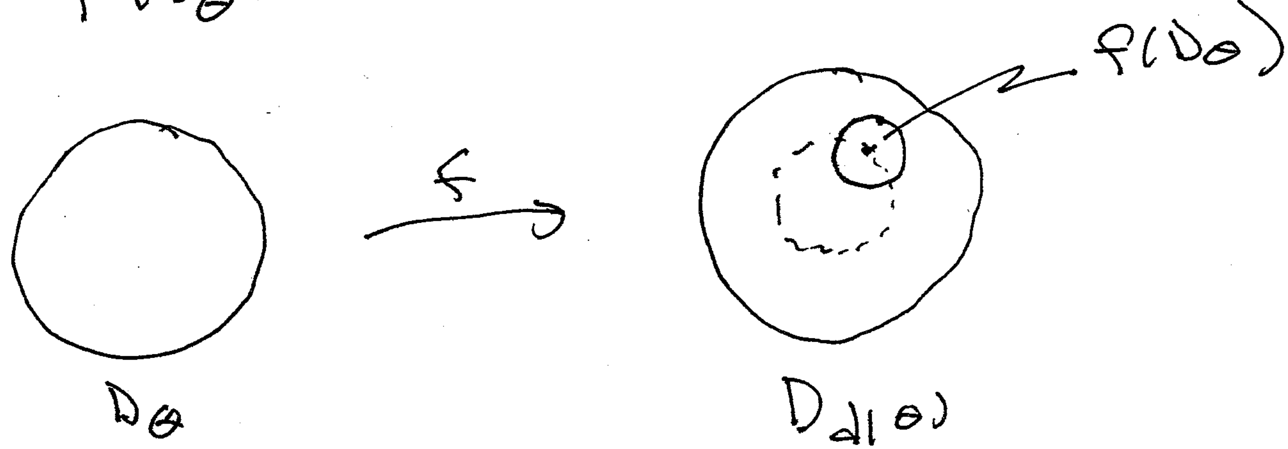
- First, T is connected and $T \supseteq f(T) \supseteq f^2(T) \supseteq \cdots$ so $\Lambda = \bigcap_{n=1}^{\infty} f^n(T)$ is the nested intersection of connected sets and thus is connected.

- We return now to the analysis of Λ via the disks $D_\theta = \{\theta\} \times D^2$

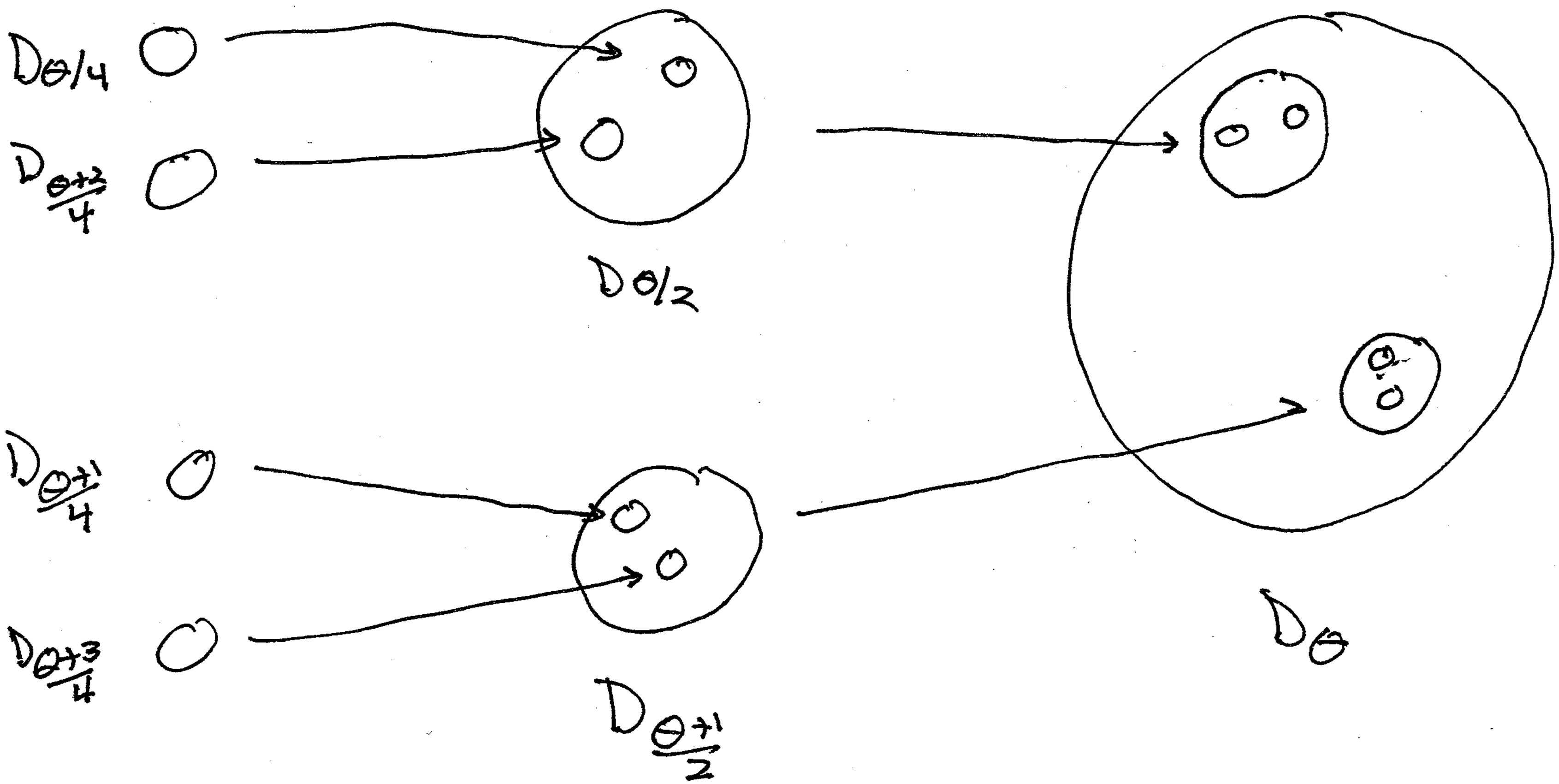


- Since $f : (\theta, z) \to \left(d(\theta), \frac{1}{10} z + \frac{1}{2} e^{2\pi i \theta}\right)$

$f(D_\theta) \subseteq D_{d(\theta)}$



Now as last time we continue <u>backwards</u>



↑ these are drawn smaller to fit them in

- Thus if $\theta, \theta_1, \theta_2, \ldots$ is a sequence with $d(\theta_{n+1}) = \theta_n$

$$D_\theta \supseteq f(D_{\theta_1}) \supseteq f^2(D_{\theta_2}) \supseteq \cdots$$

and since $\operatorname{diam} f^n(D_{\theta_2}) = \frac{1}{10^n} \operatorname{diam}(D_\theta) = 1/10^n$

each point in $\Omega \cap D_{\theta_0}$ is expressed uniquely by some $\bigcap_{n=0}^{\infty} f^n(D_{\theta_n})$ where $\theta_0, \ldots, \theta_n, \ldots$

is such that $d(\theta_{n+1}) = \theta_n$.

- Now for each θ, $d^{-1}(\theta)$ consists of exactly two points $\frac{\theta}{2}$ and $\frac{\theta}{2} + \frac{1}{2} \bmod 1$
- Thus $\Omega \cap D_{\theta_0}$ is a binary Cantor set

- Now – let $D(\theta_1, \theta_2) = (\theta_1, \theta_2) \times D^2$, $0 < \theta_2 - \theta_1 < \frac{1}{100}$

and $\{I_n\}$ be a sequence of intervals in S^1
with $d(I_0) = (\theta_1, \theta_2)$ and $d(I_{n+1}) = I_n$ $\forall n$.

- Note that d^{-1} shrinks intervals by $1/2$ and so
$|I_n| < \frac{1}{100} \frac{1}{2^n}$

- Then as with the single point case, a ~~piece~~ connected component of $\Lambda \cap D(\theta_1, \theta_2)$ is given by

$$\bigcap_{n=0}^{\infty} f^n(D(I_n))$$

$D(\theta_1, \theta_2)$

- This is an interval (needs a little work)
and so $D(\theta_1, \theta_2)$ is Cantor set cross interval.

- For the dynamics results we need two lemmas

- **Lemma 1**: The periodic points of $d: S^1 \to S^1$ are dense in S^1

<u>Proof</u>: We saw this already using the semi conjugacy from (Σ_2^+, σ) to (S^1, d), but here is a direct proof.

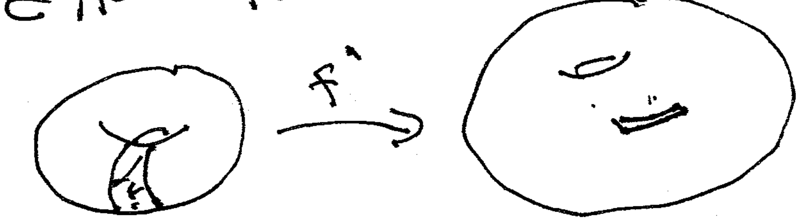
Fix $n > 0$. If $d^n(\theta) = \theta \Leftrightarrow 2^n \theta = \theta \bmod 1 \Leftrightarrow$

$2^n \theta = \theta + k$ for some $0 \leq k < 2^n \Leftrightarrow$

$\theta = \dfrac{k}{2^n - 1}$. Now the set $\left\{ \dfrac{k}{2^n - 1} \right\}_{0 \leq k < 2^n}$

is equally distributed around S^1 with separation $1/(2^n - 1)$. This is true for each n, and $2^n - 1 \to \infty$ and so periodic points are dense.

<u>Lemma 2</u>: The open sets $\{ f^n(D(\theta_1,\theta_2)) \cap \Lambda \}$ over all $0 < \theta_2 - \theta_1 < 1/10$, $n \in \mathbb{N}$ form a base for the topology of Λ.



<u>Proof</u> By construction

<u>NOTE</u>: We will be sloppy and just write $f^n(D(\theta_1,\theta_2))$ for $f^n(D(\theta_1,\theta_2)) \cap \Lambda$

<u>Proof</u> that periodic points are dense for $f|_\Lambda$

Let θ_0 be such that $d^n(\theta_0) = \theta_0$ and so $f^n(D_{\theta_0}) \subseteq D_{\theta_0}$. But D_{θ_0} is a disk, so by the Brouwer fixed point theorem

f^n has a fixed point in D_{Θ_0} i.e. f has a periodic point in D_{Θ_0}. Since the periodic points of d are dense in S^1, we have that

$$\{\Theta : D_\Theta \text{ contains a periodic point of } f\}$$

is dense in S^1.

Now to show periodic points of $f|\Lambda$ are dense in Λ, by Lemma 2, it suffices to show that for all n, θ_1, θ_2 that $f^n(D(\theta_1,\theta_2))$ contains a periodic point. But, since (θ_1,θ_2) is open we know there is a $\Theta_0 \in (\Theta_1,\theta_2)$ such that D_{Θ_0} contains a periodic point p. Thus $f^n(p) \in f^n(D_{\Theta_0}) \subseteq f^n(D(\theta_1,\theta_2))$ and $f^n(p)$ is a periodic point since p was. ∎

<u>Proof</u> of transitivity of $f|_{\perp}$: Again using Lemma 2 and a result earlier in the course, we must show that given two open sets $f^n(D(\theta_1,\theta_2))$ and $f^{n'}(D(\theta_1',\theta_2'))$

There is a $k>0$ with

$$f^k\left(f^n(D(\theta_1,\theta_2))\right) \wedge f^{n'}(D(\theta_1',\theta_2')) \neq \phi$$

Now let K_1 be such that

$$2^{k_1+n}(\theta_2-\theta_1) > 1$$

Thus $f^{K_1+n}(D(\theta_1,\theta_2)) \wedge D(\theta_1',\theta_2') \neq \phi$

and so $f^{K_1+n+n'}(D(\theta_1,\theta_2)) \wedge f^{n'}(D(\theta_1',\theta_2')) \neq \phi$

so $K = K_1+n'$ works ■

Proof of sensitive dependence

Given $p \in \Lambda$ and $\varepsilon > 0$ we show $\exists\, p'$ with $\underline{d}(p, p') < \varepsilon$ and $n \geq 0$ with $\underline{d}(f^n(p), f^n(p')) \geq 1/4$

Say p is given by $p = \bigcap_{n=0}^{\infty} f^n(D_{\theta_n})$.

Now pick $\theta_0', \theta_1', \ldots$ with θ_0' close to θ_0 and for each n, pick the inverse θ_{n+1}' of θ_n' so that θ_{n+1}' is close to θ_{n+1} and d^{-1} shrinks (each $d^{-1}(\theta)$ consists of two points distances by $1/2$

By picking θ_0' close enough to θ_0 we can ensure that $p' = \bigcap_{n=0}^{\infty} f^n(D_{\theta_n'})$ satisfies

$\underline{d}(p, p') < \varepsilon$ (we will do this more carefully next lecture)

The underlined d's are the distance in \Lambda

Now pick n so that $1/4 < 2^n | \theta_0 - \theta_0' | < 1/2$

then $f^n(p) \in D_{d^n(\theta_0)}$ and $f^n(p') \in D_{d^n(\theta_0')}$

and so $d(f^n(p), f^n(p')) \geq | d^n(\theta_0), d^n(\theta_0') | > 1/4$. ∎

Proof that Λ is compact and not path connected

- Λ is the intersection of a nested family of compact sets and thus is compact.
- Fix some D_θ. By construction any injective path in Λ that intersects D_θ, must go around once before intersecting it again. Thus any path can hit D_θ finitely many times. But $\Lambda \cap D_\theta$ is a Cantor set, and thus is uncountable, and so Λ has uncountably many path components ∎