Inverse Limits and the Solenoid

- In analysing the solenoid we encountered sequences $\theta_0\, \theta_1\, \theta_2 \ldots$ where $d(\theta_{n+1}) = \theta_n$ and $d: S^1 \to S^1$, $d(\theta) = 2\theta \bmod 1$
- This is a special case of a construction called the inverse limit that is used in various areas of mathematics
- We just consider the dynamical/topological version

- Let X be a compact, metric space and $f : X \to X$ be continuous and onto. Also, X is perfect, every point is limit point.

- A thread is a sequence $\underline{x} = x_0 x_1 x_2 \ldots$ with $f(x_{n+1}) = x_n$

- The inverse limit is the collection of threads.

- More formally,

$$\underleftarrow{\lim}(X, f) = \{ \underline{x} \in X^{\mathbb{N}} : f(x_{n+1}) = x_n, \forall n \}$$

- The topogy on $X^{\mathbb{N}} = \{ \underline{x} : x_n \in X, \forall n \}$ is given by the metric

$$\hat{d}(\underline{x}, \underline{y}) = \sum_{n=0}^{\infty} \frac{d(x_n, y_n)}{2^n}$$

where d is the metric on X

- Since X is compact, it has a finite diameter ie. a max $d(x,y)$, and so $\hat{d}$ converges.

- The Tychonoff theorem says that $X^{\mathbb{N}}$ is compact.
- Using the continuity of f, it is easy to check that $\varprojlim (X, f)$ is closed in $X^{\mathbb{N}}$ and therefore compact.
- Notation: If f is understood we just write $\hat{X} = \varprojlim (X, f)$
- FACT: If X is connected so is $\hat{X}$.
- The projections $\pi_n : \hat{X} \to X$ are given by $\pi_n(\underline{x}) = x_n$.

- Now define $\hat{f} : \hat{X} \to \hat{X}$ via (the shift or natural extension)

$$\hat{f}(x_0, x_1, \ldots, x_n, \ldots) = (f(x_0), x_0, x_1, \ldots)$$

- <u>FACT</u>: $\hat{f}$ is a homeomorphism with inverse

$$\hat{f}^{-1}(x_0, x_1, \ldots) = (x_1, x_2, x_3, \ldots)$$

- <u>Proof</u>: A useful lemma: If $g : Y \to Z$ is bijective, continuous and Y is compact $\Rightarrow g$ is a homeomorphism

So Assume $\underline{x}^j \to \underline{x}^0$, then obviously $\hat{f}^{-1}(\underline{x}^j) \to \hat{f}^{-1}(\underline{x}^0)$ and so $\hat{f}^{-1}$ is continuous and $\hat{X}$ is compact, so $\hat{f}$ is a homeo.

& it is obvious that $id = \hat{f}^{-1}\hat{f} = \hat{f}\hat{f}^{-1}$

- We have a semiconjugacy

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{f}"] \arrow[d, "\pi_0"'] & \hat{X} \arrow[d, "\pi_0"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

Proof $\pi_0 \hat{f}(\underline{x}) = \pi_0(f(x_0), x_0, \dots) = f(x_0)$

$f \pi_0(\underline{x}) = f(x_0)$.

- New terminology: Given a semiconjugacy $(\hat{X}, \hat{f})$ is called an <u>extension</u> of (X, f) and (X, f) is called a factor of $(\hat{X}, \hat{f})$

- A minimality property of $(\hat{X}, \hat{f})$

If (Y, g) with g a homeomorphism is an extension of (X, f) then $(\hat{X}, \hat{f})$ is a factor of (Y, g). In other words, $(\hat{X}, \hat{f})$ is the smallest homeomorphism extension of (X, f)

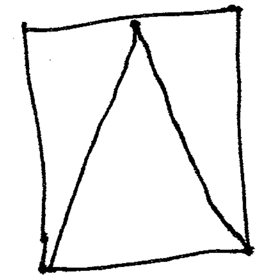
Given diagram, $\exists\, \beta$

$$\begin{tikzcd}
(Y, g) \arrow[r, dashed, "\beta"] \arrow[dr, "\alpha"'] & (\hat{X}, \hat{f}) \arrow[d, "\pi_0"] \\
& (X, f)
\end{tikzcd}$$

Proof: Let $\beta : Y \to \hat{X}$ be defined by $\beta(y) = \alpha(y), \alpha(g^{-1}(y)), \alpha(g^{-2}(y)), \ldots$ and check it has the desired properties.

<u>example</u>: If $X = [0,1]$ and $f: X \to X$

is $f(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ -2x+2 & 1/2 \le x \le 1 \end{cases}$

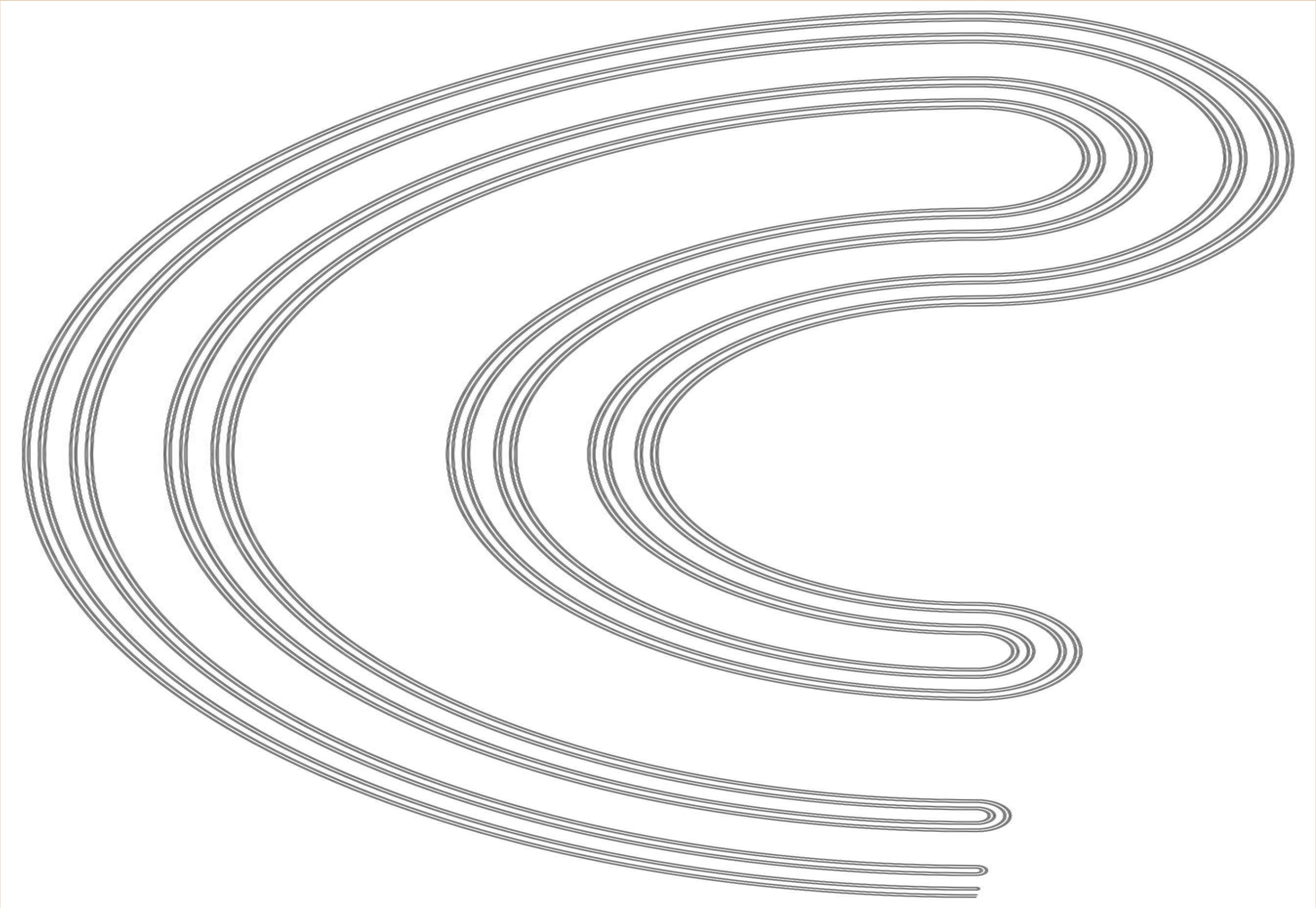


full tent map

Then $\varprojlim (X, f)$ is the Knaster continuum

(bucket handle, unstable manifolds in Horseshoe).

Knaster-Bucket handle-Horseshoe



- The dynamics of $(\hat{X}, \hat{f})$ are closely related to those of (X, f).

- Recall our many objects are compact invariant sets.

- Assume $Z \subseteq X$ is compact, invariant under f

- Let $\hat{Z} \subseteq \hat{X}$ be defined by

$$\hat{Z} = \{ \underline{x} \in \hat{X} : x_n \in Z \text{ for all } n \}$$

- **Lemma**: $\hat{Z}$ is a compact, $\hat{f}$ invariant set

Proof If $\underline{z} = z_0 z_1 \ldots$, $\underline{z} \in \hat{Z}$ $\Rightarrow$ $\hat{f}(\underline{z}) = f(z_0), z_0, \ldots \in \hat{Z}$ since $f(z_0) \in Z$

$\hat{f}(\underline{z}) = (z_1, z_2, \ldots) \in Z$ and also under the inverse.

So $\hat{Z}$ is $\hat{f}$-invariant

Since $\hat{\underline{X}}$ is compact, it suffices to show $\hat{Z}$ is closed, if $\underline{z}^{(i)} \in \hat{Z}$ and $\underline{z}^{(i)} \to \underline{z}^{(0)}$ Then looking component wise,

for each n, $z_n^{(i)} \to z_n^{(0)}$ ———..

and so $z_n^{(0)} \in Z$ since Z is compact but

Now we also need to check that $\underline{z}^{(0)} \in \hat{\underline{X}}$

That follows since $\hat{\underline{X}}$ is closed. $\boxtimes$

<u>Example</u>: If $Z = \{z, f(z), .., f^{n-1}(z)\}$ with $f^n(z) = z$ $\Rightarrow$ $\hat{Z}$ is a periodic orbit

$\underline{z} = z, f^{n-1}(z), .., f(z), z, f^{n-1}(z), ...$ also period n.

and $\hat{f}^i(\underline{z})$ for $i = 1, .., n-1$.

<u>Lemma</u> If (Z, f) is minimal in (X, f) then $(\hat{Z}, \hat{f})$ is minimal in $(\hat{X}, \hat{f})$

<u>Lemma</u> If (X, f) is transitive then so is $(\hat{X}, \hat{f})$

<u>Lemma</u>: If (Σ, f) has dense periodic points so does $(\hat{\Sigma}, \hat{f})$

<u>Note</u>: converses are true since

$$\begin{tikzcd} \hat{\Sigma} \arrow[r, "\hat{f}"] \arrow[d, "\pi_0"'] & \hat{\Sigma} \arrow[d, "\pi_0"] \\ \Sigma \arrow[r, "f"] & \Sigma \end{tikzcd}$$

is a semi conjugacy

$\boxed{\text{The simple solenoid}}$

$\mathbb{X} = S^1 \quad d: \mathbb{X} \to \mathbb{X} \quad d(\theta) = 2\theta \mod 1$

(or in complex notation $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$, $d(z) = z^2$)

$\Gamma = \varprojlim (S^1, d)$ is the solenoid

So $\Gamma = \{ \theta_0, \theta_1, \ldots : \theta_i \in S^1, d(\theta_{i+1}) = \theta_i, \forall i \}$

Since (S^1, d) is transitive, has dense periodic points, sensitive dependence on initial condition

then so does $(\Gamma, \hat{d})$.

The solenoid is a topological group

- This is easiest in complex notation. $S^1 = \{|z| = 1\}$ is a multiplicative group. $z_1 z_2 \in S^1$ when z_1 and z_2 are.
  $1/z_1 = \bar{z}_1 \in S^1$ when z_1 is.
- Then $d : S^1 \to S^1$ is $z \mapsto z^2$
- For $\underline{z} = z_0, z_1, \ldots$ in Γ
  $\underline{y} = y_0, y_1, \ldots$

  let $\underline{z} * \underline{y} = z_0 y_0, z_1 y_1, z_2 y_2, \ldots$
  and note $d(z_{n+1} y_{n+1}) = (z_{n+1} y_{n+1})^2 = z_{n+1}^2 y_{n+1}^2 = z_n y_n$

  so $\underline{z} * \underline{y} \in \Gamma$, as is $\underline{z}^{-1}$.
- Next time: attractor Λ is topologically conjugate to $(\Gamma, \hat{d})$.