Inverse limits and the Solenoid attractor, cont.

X is compact metric, perfect $f: X \to X$ is onto

$\hat{X} = \varprojlim (X, f) = \{ \underline{x} \in X^{\mathbb{N}} : f(x_{n+1}) = x_n \ \forall n \}$

$\pi_n : \hat{X} \to X$, $\pi_n(\underline{x}) = x_n$ projections

$\hat{f}(x_0, x_1, \ldots) = f(x_0), f(x_1), \ldots$

<u>Lemma</u> The collection of all $\pi_n^{-1}(B_\varepsilon(x))$ for $n \in \mathbb{N}$, $B_\varepsilon(x)$ an open ball in X form a base for the topology of $\hat{X}$

<u>Theorem</u>: If (X, f) is transitive, so is $(\hat{X}, \hat{f})$

<u>Proof of theorem</u>. First note the preliminary easy to check fact that when $a \geq b$, $\pi_b(\hat{f}^a(\underline{x})) = f^{a-b}(x_0)$ in general.

Now for the main proof, say $O^+(x_0, f)$ is dense in X. Since f is onto we may find an element $\underline{x} = x_0 x_1 \ldots \in \hat{X}$. Given any basic open set $\pi_n^{-1}(B_\varepsilon(y))$ we must produce some m with $\hat{f}^m(\underline{x}) \in \pi_n^{-1}(B_\varepsilon(y))$

Since $O^+(x_0, f)$ is dense, $\exists k$ with $f^k(x_0) \in B_\varepsilon(y)$

and then using the preliminary

$$\pi_n(\hat{f}^{n+k}(\underline{x})) = f^k(x_0) \in B_\varepsilon(y)$$

and so $\hat{f}^{n+k}(\underline{x}) \in \pi_n^{-1}(B_\varepsilon(y))$ as required. ∎

Γ and Λ

Back to the solenoid attractor $f: S^1 \times D^2 \to S^1 \times D^2$

$$f(\theta, z) = \left(d(\theta), \frac{1}{10} z + \frac{1}{2} e^{2\pi i \theta}\right)$$

and $\Lambda = \bigcap_{n=0}^{\infty} f^n(T)$

Theorem $(\Lambda, f|_\Lambda)$ is topologically conjugate to $(\varprojlim (S^1, d), \hat{d})$

Proof: As above let $\Gamma = \varprojlim (S^1, d)$

Let $\pi: S^1 \times D^2 \to S^1$ be projection on the first coordinate, $\pi(\theta, z) = \theta$ and $\alpha: \Lambda \to \Gamma$ is defined by $(\alpha(p))_i = \pi(f^{-i}(p))$. Now from the formula for f $\pi \circ f = d \circ \pi$ on Λ

(which is just to say that f acts like d in the 1^{st} coord.)

so

and $d\left(\pi\left(f^{-i}(p)\right)\right)$

$= \pi\left(f f^{-i}(p)\right) = \pi\left(f^{-i+1}(p)\right) = (\alpha(p))_{i-1}$

and so as claimed $\alpha(p) \in \Gamma = \varprojlim(S^1, d)$.

We show that $\alpha : \Lambda \to \Gamma$ is a homeomorphism

and

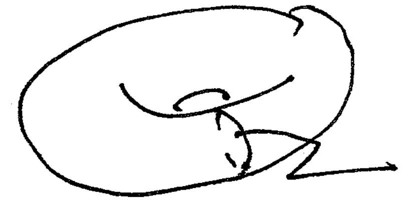
$$\begin{tikzcd} \Lambda \arrow[r, "f|_\Lambda"] \arrow[d, "\alpha"'] & \Lambda \arrow[d, "\alpha"] \\ \Gamma \arrow[r, "\hat{d}"] & \Gamma \end{tikzcd}$$

commutes.

(1) α is bijective: Recall that each point $p \in \Lambda$ is described uniquely as

$$p = \bigcap_{n=0}^{\infty} f(D_{\theta_n}) \quad \text{where } d(\theta_{n+1}) = \theta_n$$

or $\theta_0, \theta_1, \ldots \in \varprojlim(S^1, d)$

and $D_\theta = \{\theta\} \times D^2$  D_θ

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define $\beta : \Gamma \to \Lambda$ as

$$\beta(\theta_0, \theta_1, \ldots) = \bigcap_{n=0}^{\infty} f(D_{\theta_n})$$

and it is immediate that $\alpha \circ \beta = id$, $\beta \circ \alpha = id$

and so α is bijective.

(2) bi-continuity. Recall $\{ f^n(D(\theta,\theta')) \}$ over all n and (θ,θ') and open interval in S^1 form a base for the topology of Λ. Also, $\{ \pi_n^{-1}((\theta,\theta')) \}$ over all n form a base for the topology of Γ

and $\alpha \mid f^n(D(\theta,\theta')) = \pi_n^{-1}((\theta,\theta'))$

and so α is continuous. For β,

$$\beta(\pi_n^{-1}((\theta,\theta')) = f^n(D(\theta,\theta'))$$

[Recall we are using a short hand $f^n(D(\theta,\theta'))$

$$= f^n(D(\theta,\theta')) \cap \Lambda]$$

(3) diagram commutes, we check the i^{th} coordinate

$(\alpha \circ f(p))_i = \pi(f^{-i}(f(p))) = \pi f^{-i+1}(p) = d\pi(f^{-i}(p))$

since $\pi f = d\pi$ from above

$$(\hat{d} \circ \alpha(p))_i = \left(\hat{d}\,(\pi(p), \pi(f^{-1}(p)), \ldots, \pi f^{-i}(p), \ldots\right)_i$$

$$= (d\pi(p), d\pi(f^{-1}(p)), \ldots, d\pi f^{-i}(p), \ldots)_i$$

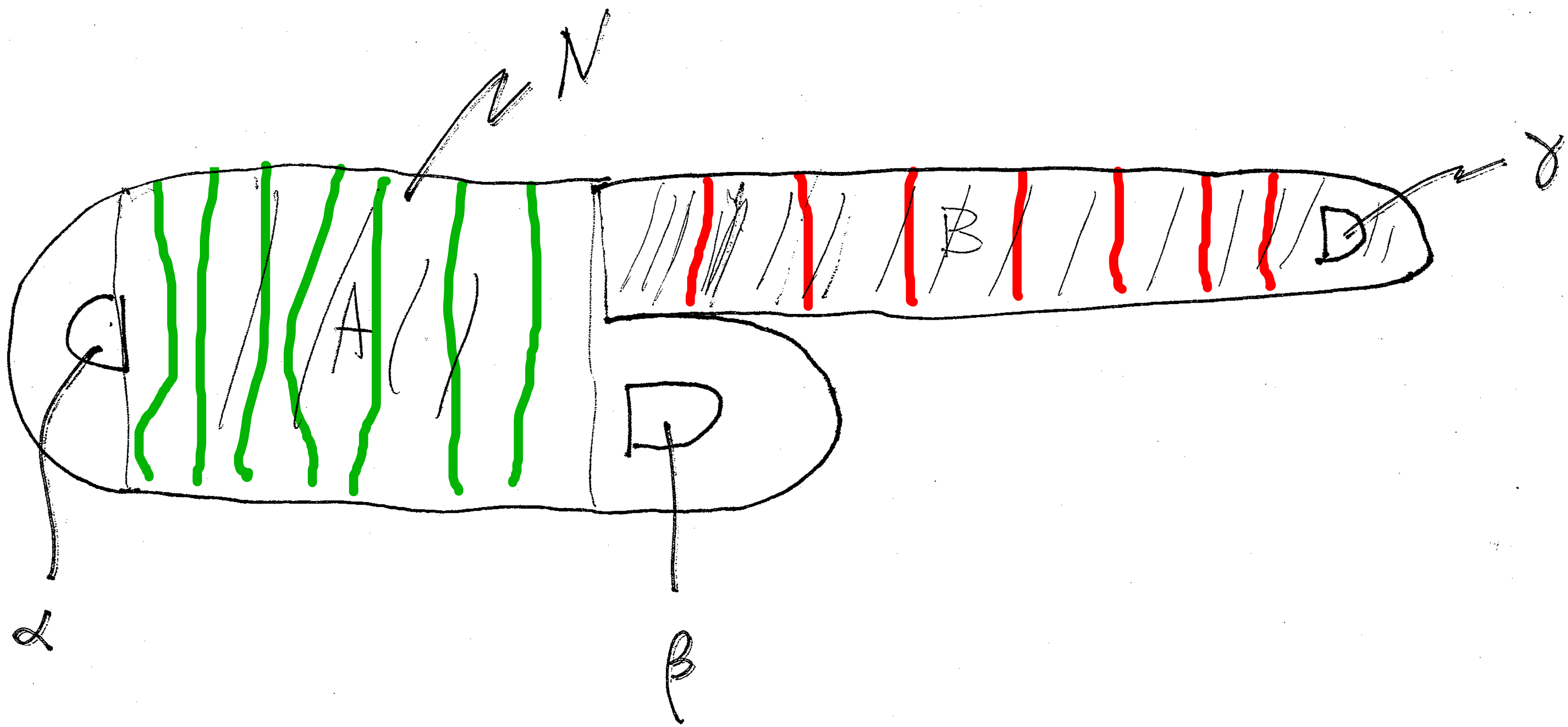
$$= d\pi f^{-i}(p)$$

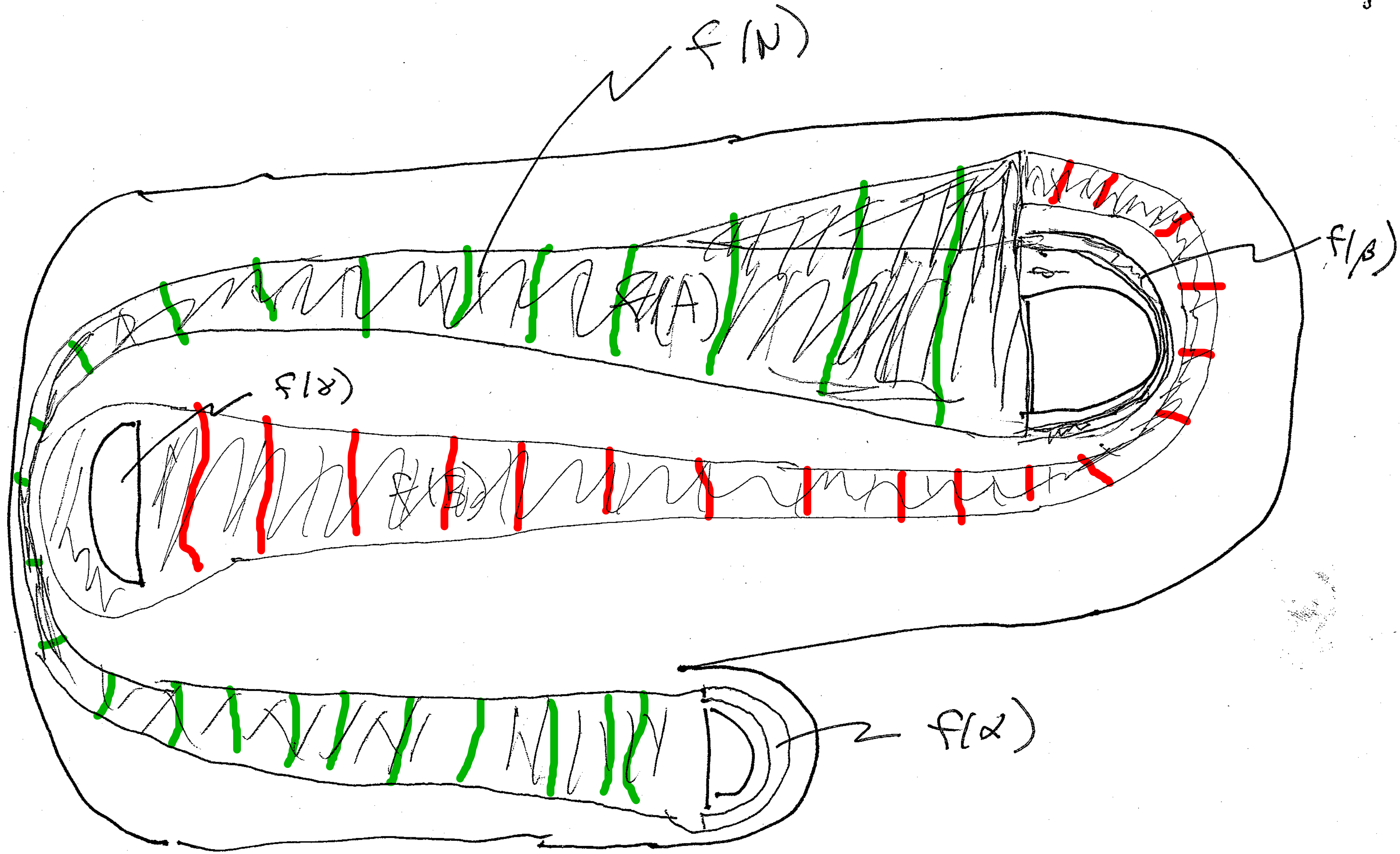
∎

- Though we proved (or noted) it directly above, since we know $d: S^1 \to S^1$ has dense periodic orbits and is transitive, then so is $\Gamma = \varprojlim(S^1, d)$ under $\hat{d}$ and thus $(\Lambda, f|_\Lambda)$ since its topologically conjugate is $(\Gamma, \hat{d})$

- This is just one case of a class of attractors described by inverse limits.

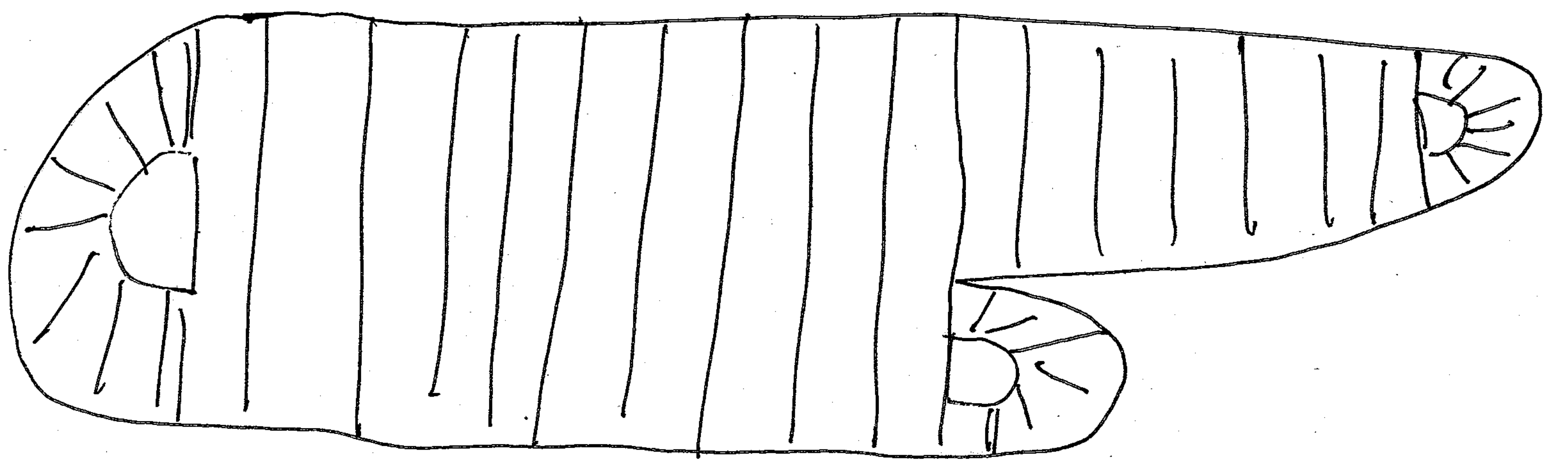
- We briefly discuss the Plykin attractor


N
A
B
α
β
γ

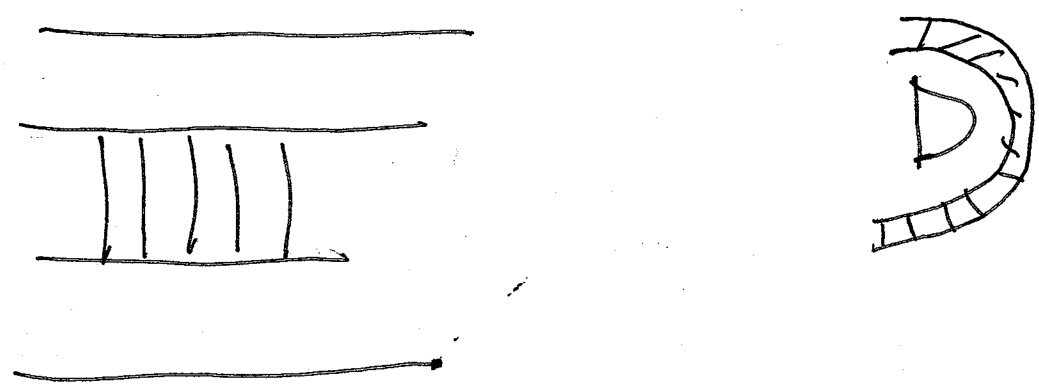


f(N)
f(β)
f(A)
f(γ)
f(B)
f(α)

More carefully, foliate N with segments



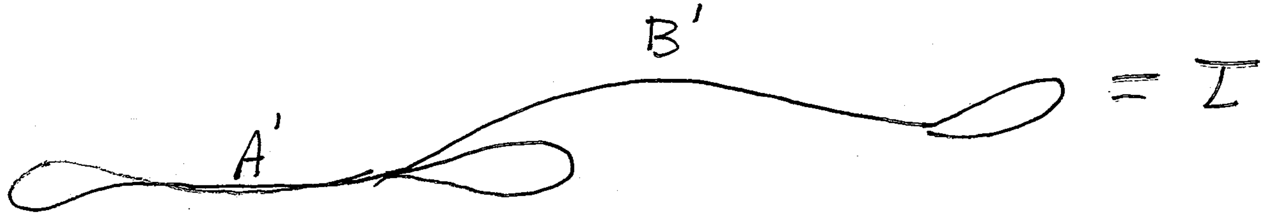
and send $f(N) \subseteq N$ taking segments to segments



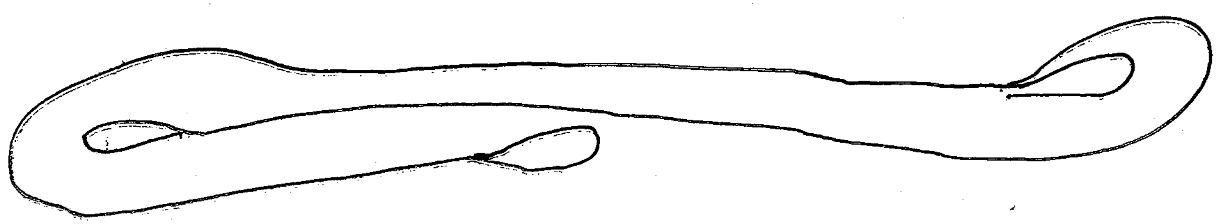
The attractor (Plykin) is $\Lambda = \bigcap_{n=0}^{\infty} f^n(N)$

- So to understand the map we just need to see its action on the "space of segments"

- We construct this space by collapsing down segments to a point yielding



its image is



collapsed back down to I

Call this map $g: I \to I$

Then just like with the solenoid

$(\Lambda, f|_\Lambda)$ is topologically conjugate
to $(\varprojlim(I, g), \hat{g})$

and properties of the inverse limit can be used to study the attractor Λ

- The attractor will be locally cantor set $\times$ interval, transitive, etc.